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THE CIRCLE

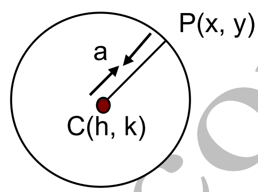
Definition

A circle is defined as the locus of a point, which moves in a plane such that its distance from a fixed point in that plane is always constant. The fixed point is called the **centre** of the circle and the constant distance is called the **radius** of the circle.

Standard Equation of a circle

To find the equation of any circle whose centre and radius are given.

Let C be the centre of the circle and its coordinates be (h, k). Let the radius of the circle be a and let P (x, y) be any point on the circumference. Then



$|CP| = a$

$\Rightarrow \quad CP^2 = a^2$

$\Rightarrow \quad (x - h)^2 + (y - k)^2 = a^2$

This is the relation between the coordinates of any point on the circumference and hence it is the required equation of the circle having centre at (h, k) and radius equal to a.

Note

(i) The above equation is known as the central form of the equation of a circle.

(ii) If the centre of the circle is at the origin and radius is a, then from the above form the equation of the circle is

$$x^2 + y^2 = a^2$$

Illustration 1

Find the equation of a circle whose centre is (2, –3) and radius 5.

The equation of the required circle is

$$(x - 2)^2 + (y + 3)^2 = 5^2 \quad \Rightarrow \quad x^2 + y^2 - 4x + 6y - 12 = 0$$

Illustration 2

Find the equation of a circle whose radius is 6 and the centre is at the origin.

The equation of the required is

$$x^2 + y^2 = 6^2 \quad \Rightarrow x^2 + y^2 = 36.$$

Some particular cases of the central form of the equation of a circle

The equation of a circle with centre at (h, k) and radius equal to a, is

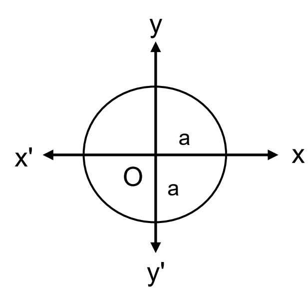
$$(x - h)^2 + (y - k)^2 = a^2 \qquad \text{...(i)}$$

(i) When the centre of the circle coincides with the origin

i.e. h = k = 0.

Then the equation (i) reduces to

$$x^2 + y^2 = a^2.$$



(ii) When the circle passes through the origin

Let O be the origin and C(h, k) be the centre of the circle.

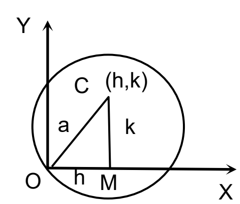
Draw CM $\perp$ OX.

In ΔOCM, $OC^2 = OM^2 + CM^2$

i.e., $a^2 = h^2 + k^2$

The equation of the circle (i) then becomes

$$(x - h)^2 + (y - k)^2 = h^2 + k^2 \qquad \text{or} \qquad x^2 + y^2 - 2hx - 2ky = 0.$$



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(iii) When the circle touches x-axis

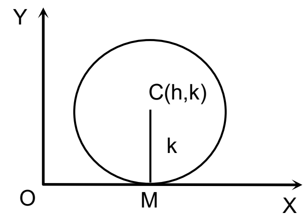
Let C (h, k) be the centre of the circle. Since the circle touches the x-axis,

$\therefore$ a = k

Hence the equation of the circle is

$$(x - h)^2 + (y - a)^2 = a^2$$

or $x^2 + y^2 - 2\,hx - 2\,ay + h^2 = 0$



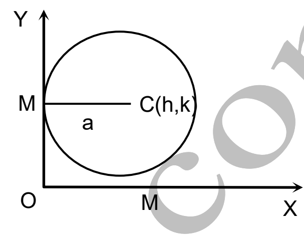
(iv) When the circle touches y-axis

Let C (h, k) be the centre of the circle. Since the circle touches the y-axis,

$\therefore$ h = a

Hence the equation of the circle is $(x - a)^2 + (y - k)^2 = a^2$

or $x^2 + y^2 - 2ax - 2ky + k^2 = 0$

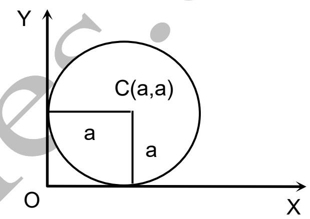


(v) When the circle touches both the axes

In this case h = k = a

Hence the equation of the circle is $(x - a)^2 + (y - a)^2 = a^2$

or $x^2 + y^2 - 2ax - 2ay + a^2 = 0$.



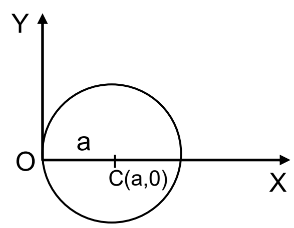
(vi) When the circle passes through the origin and centre lies on x-axis

In this case k = 0 and h = a.

Hence the equation of the circle is

$$(x - a)^2 + (y - 0)^2 = a^2$$

or $x^2 + y^2 - 2ax = 0$.



(vii) When the circle passes through origin and centre lies on y-axis

In this case h = 0 and k = a.

Hence the equation of the circle is $(x - 0)^2 + (y - a)^2 = a^2$

or $x^2 + y^2 - 2ay = 0$.

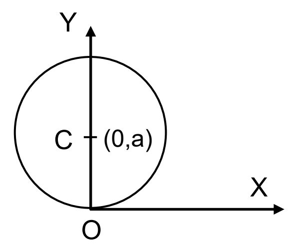


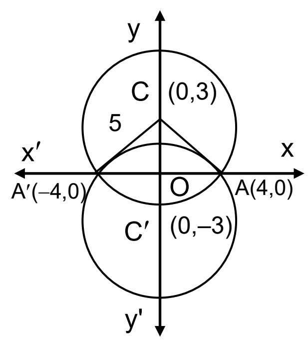
Illustration 3

Find the equations of the circle which passes through two points on the x-axis which are at distance 4 units from the origin and whose radius is 5.

As is evident from figure there are two circles which pass through two points A and A′ on x-axis which are at a distance 4 from the origin. The centres of these circles lie on y-axis.

In ΔOAC, $AC^2 = OA^2 + OC^2$

$\Rightarrow \quad 5^2 = 4^2 + OC^2 \Rightarrow OC = 3$.



So, the coordinates of the centres of the required circles are C (0, 3) and C′ (0, –3). Hence the equation of the required circles are

$$(x + 0)^2 + (y \pm 3)^2 = 5^2 \quad \text{or} \quad x^2 + y^2 \pm 6y - 16 = 0$$

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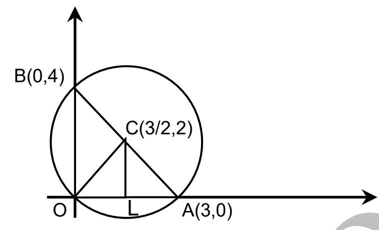
Illustration 4

Find the equation of the circle, which passes through the origin and cuts off intercepts 3 and 4 from the positive parts of the axes respectively.

We have OA = 3, OB = 4.



$\therefore$ OL = $\frac{3}{2}$ and CL = 2.

In ΔOLC, $OC^2 = OL^2 + LC^2$

$\Rightarrow$ $OC^2 = \left(\frac{3}{2}\right)^2 + 2^2$

$\Rightarrow$ $OC = \frac{5}{2}$.

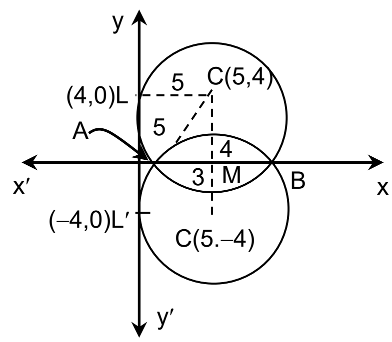
Thus, the required circle has its centre at $\left(\frac{3}{2}, 2\right)$ and radius $\frac{5}{2}$.

Hence, its equation is $\left(x - \frac{3}{2}\right)^2 + (y - 2)^2 = \left(\frac{5}{2}\right)^2$

Illustration 5

Find the equation of a circle which touches y-axis at a distance of 4 units from the origin and cuts an intercept of 6 units along the positive direction of x-axis.

The given circle touches y-axis at (0, 4) and cuts an intercept AB = 6 along the positive direction of x-axis. As shown in figure there are two such circles.



In ΔCMA, we have

$CA^2 = CM^2 + AM^2$

$\Rightarrow$ $CA^2 = 4^2 + 3^2 \Rightarrow CA = 5$.

Also, CL = CA = 5

Thus, the coordinates of the centre are (5, 4) or (5, –4) and radius = 5

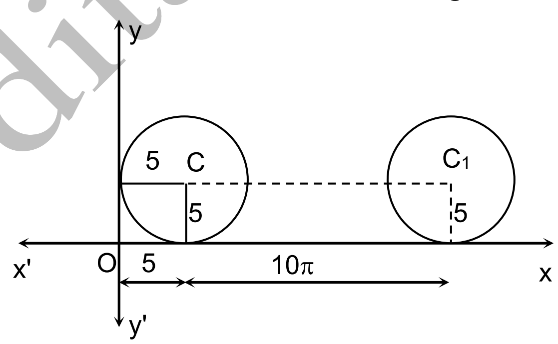
Hence the equation of the required circles are

$(x - 5)^2 + (y \pm 4)^2 = 5^2$ or $x^2 + y^2 - 10x \pm 8y + 16 = 0$.

Illustration 6

A circle of radius 5 units touches the coordinate axes in the first quadrant. If the circle makes one complete roll on x-axis along the positive direction of x-axis, find its equation in new position.

Let C and C_1 be the centres of the circle in its initial and final positions. The, coordinates of C are (5, 5). In making one complete roll on x-axis, the centre C moves through the distance



CC_1 = Circumference of the circle = 10π

So, the coordinates of the centre of the circle in the new position are $(5 + 10\pi, 5)$.

Radius of the circle in its new position is 5 units. Hence, its equation is $\{x - (5 + 10\pi)\}^2 + (y - 5)^2 = 5^2$

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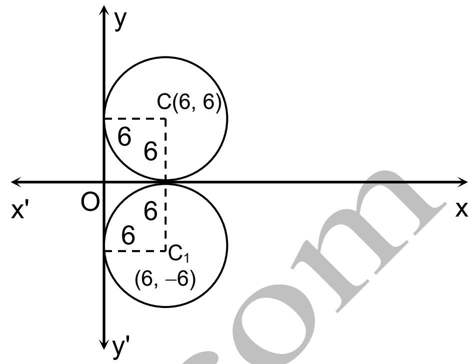
Illustration 7

A circle of radius 6 units touches the coordinate axes in the first quadrant. Find the equation of its image in the line mirror y = 0.



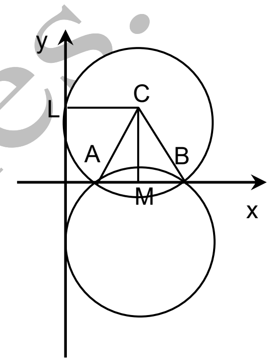
The given circle has radius 6 and the coordinates of its centre C are (6, 6).
The coordinates of its image C_1 in the line mirror y = 0 i.e. x –axis are (6, –6).
So, the centre of the required circle is at $C_1(6, –6)$ and its radius is 6.
Hence, its equation is

$$(x - 6)^2 + (y + 6)^2 = 6^2$$

or, $$x^2 + y^2 - 12x + 12y + 36 = 0$$

General Equation of a Circle

Prove that the equation $x^2 + y^2 + 2gx + 2fy + c = 0$ always represents a circle whose centre is (–g, –f) and radius = $\sqrt{g^2 + f^2 - c}$.



PROOF. The given equation is $x^2 + y^2 + 2gx + 2fy + c = 0$ ...(i)

$\Rightarrow$ $(x^2 + 2gx + g^2) + (y^2 + 2fy + f^2) = g^2 + f^2 - c$

$\Rightarrow$ $(x + g)^2 + (y + f)^2 = \left(\sqrt{g^2 + f^2 - c}\right)^2$

$\Rightarrow$ $(x - (-g))^2 + (y - (-f))^2 = \left(\sqrt{g^2 + f^2 - c}\right)^2$

This is of the form $(x - h)^2 + (y - k)^2 = a^2$ which represents a circle having centre at (h, k) and radius equal to a.
Hence, the given equation (i) represents a circle whose centre is (–g, –f) i.e.

$$\left(-\frac{1}{2}\text{coeff. of } x, -\frac{1}{2}\text{ coeff. of } y\right) \text{ and}$$

Radius = $\sqrt{g^2 + f^2 - c}$ = $\sqrt{\left(\frac{1}{2}\text{coeff. of } x\right)^2 + \left(\frac{1}{2}\text{coeff. of } y\right)^2 - \text{constant term}}$

Note :

1. The equation $x^2 + y^2 + 2gx + 2fy + c = 0$ represents a circle of radius $\sqrt{g^2 + f^2 - c}$.
If $g^2 + f^2 - c > 0$ then the radius of the circle is real and hence the circle is also real.
If $g^2 + f^2 - c = 0$ then the radius of the circle is zero. Such a circle is known as a point circle.
If $g^2 + f^2 - c < 0$ then the radius $\sqrt{g^2 + f^2 - c}$ of the circle is imaginary but the centre is real. Such a circle is called an imaginary circle as it is not possible to draw such a circle.

2. Special features of the general equation $x^2 + y^2 + 2gx + 2fy + c = 0$ of the circle are :
- (i) It is quadratic in both x and y.
- (ii) coefficient of x^2 = coefficient of y^2.
  In solving problem it is advisable to keep the coefficient of x^2 and y^2 unity.
- (iii) There is no term containing xy i.e., the coefficient of xy is zero.
- (iv) It contains three arbitrary constants viz. g, f and c.

3. The equation $ax^2 + ay^2 + 2gx + 2fy + c = 0$, $a \neq 0$ also represents a circle. This equation can also be written as

$$x^2 + y^2 + \frac{2g}{a}x + \frac{2f}{a}y + \frac{c}{a} = 0$$

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The coordinates of the centre are $\left(-\frac{g}{a}, -\frac{f}{a}\right)$ and radius = $\sqrt{\frac{g^2}{a^2}+\frac{f^2}{a^2}-\frac{c}{a}}$.

4. On comparing the general equation $x^2 + y^2 + 2gx + 2fy + c = 0$ of a circle with the general equation of second degree $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ we find that it represents a circle if a = b i.e., coefficient of x^2 = coefficient of y^2 and h = 0 i.e., coefficient of xy = 0.

5. We denote equation of a circle by $S = x^2 + y^2 + 2gx + 2fy + c = 0$. and
$S_1 = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c = 0$.

Illustration 8

Find the equation of the circle circumscribing the triangle formed by the lines x + y = 6, 2 x + y = 4 and x + 2y = 5.

Let the equations of sides AB, BC, and CA of Δ ABC are respectively

$$x + y = 6 \quad .....(i)$$
$$2x + y = 4 \quad .....(ii)$$
$$\text{and } x + 2y = 5 \quad ....(iii)$$

Solving (i) and (iii), (ii) and (iii) & (iii) and (i) we get the coordinates of A, B and C. The coordinates A, B and C are (7, –1), (–2, 8) and (1, 2) respectively.

Let the equation of the circumcircle of Δ ABC be

$$x^2 + y^2 + 2gx + 2fy + c = 0 \quad ....(iv)$$

It passes through A (7, – 1), B (–2, 8) and C (1, 2), we get

$$50 + 14g - 2f + c = 0 \quad ....(v)$$
$$68 - 4g + 16f + c = 0 \quad ....(vi)$$
$$5 + 2g + 4f + c = 0 \quad ....(vii)$$

Subtracting (v) from (vi), we get

$$18 - 18g + 18f = 0 \Rightarrow 1 - g + f = 0 \quad ....(viii)$$

Subtracting (v) from (vii), we get

$$-45 - 12g + 6f = 0 \quad ....(ix)$$

Solving (viii) and (ix) we get $g = -\frac{17}{2}, f = -\frac{19}{2}$.

Putting the values of g and f in (v), we get c = 50.

Substituting the values of g, f and c in (iv), the equation of the required circumcircle is

$$x^2 + y^2 - 17x - 19y + 50 = 0$$

Illustration 9

Find the radius of the circle $(x\cos\alpha + y\sin\alpha - a)^2 + (x\sin\alpha - y\cos\alpha - b)^2 = k^2$, if α varies, then its centre is again a circle. Also, find its centre and radius.

The given equation is

$$(x\cos\alpha + y\sin\alpha - a)^2 + (x\sin\alpha - y\cos\alpha - b)^2 = k^2$$

$\Rightarrow$ $x^2(\cos^2\alpha + \sin^2\alpha) + y^2(\sin^2\alpha + \cos^2\alpha) - 2x(a\cos\alpha + b\sin\alpha) - 2y(a\sin\alpha - b\cos\alpha) + a^2 + b^2 - k^2 = 0$

$\Rightarrow$ $x^2 + y^2 - 2x(a\cos\alpha + b\sin\alpha) - 2y(a\sin\alpha - b\cos\alpha) + a^2 + b^2 - k^2 = 0$

The coordinates of the centre of this circle are

$(a\cos\alpha + b\sin\alpha, a\sin\alpha - b\cos\alpha)$ and radius

$$= \sqrt{(a\cos\alpha + b\sin\alpha)^2 + (a\sin\alpha - b\cos\alpha)^2 - (a^2 + b^2 - k^2)}$$

$$= \sqrt{a^2(\cos^2\alpha + \sin^2\alpha) + b^2(\sin^2\alpha + \cos^2\alpha) - (a^2 + b^2 - k^2)}$$

$$= \sqrt{a^2 + b^2 - a^2 - b^2 + k^2} = k$$

Let (h, k) be the coordinates of the center of the given circle.

Then

$$h = a\cos\alpha + b\sin\alpha \text{ and } k = a\sin\alpha - b\cos\alpha$$

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To find the locus of (h, k) we have to eliminate α. Squaring and adding, we get,

$$h^2 + k^2 = (a\cos\alpha + b\sin\alpha)^2 + (a\sin\alpha - b\cos\alpha)^2$$

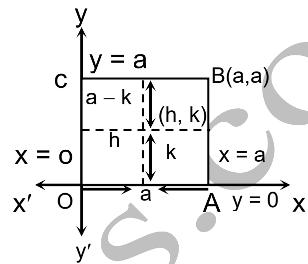
$$\Rightarrow \quad h^2 + k^2 = a^2(\cos^2\alpha + \sin^2\alpha) + b^2(\sin^2\alpha + \cos^2\alpha) \Rightarrow h^2 + k^2 = a^2 + b^2$$

So, locus of (h, k) is $x^2 + y^2 = a^2 + b^2$

This is a circle having centre at (0, 0) and radius equal to $\sqrt{a^2 + b^2}$.

Illustration 10

Prove that the locus of a point, which moves in such a way that the sum of the squares of its distances from the four sides of a square is constant (= 2 c^2), is a circle.



Let O be the origin and OABC be a square of side a whose two sides OA and OC are along OX and OY axis respectively. Then the equations of sides AB and BC are x = a and y = a respectively.

Let P (h, k) be a point such that the sum squares of its distances from the four sides of the square to $2c^2$. Then

$$k^2 + h^2 + (a - h)^2 + (a - k)^2 = 2c^2$$

$$\Rightarrow \quad 2h^2 + 2k^2 - 2ah - 2ak + 2a^2 - 2c^2 = 0$$

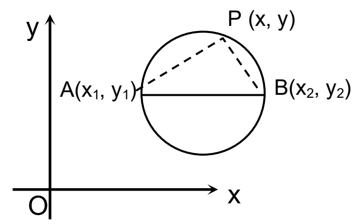
$$\Rightarrow \quad h^2 + k^2 - ah - ak + a^2 - c^2 = 0$$

Hence, locus of (h, k) is, $x^2 + y^2 - ax - ay + a^2 - c^2 = 0$, which is a circle.

Equation of a circle when end points of a diameter are given

The equation of the circle drawn on the straight line joining two given points (x_1, y_1) & (x_2, y_2) is $(x-x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$

PROOF: Let A and B be the extremities of the diameter AB having coordinates (x_1, y_1) and (x_2, y_2) respectively. Let P(x, y) be any point on the circle. Join P, A and P, B. Then,



$$m_1 = \text{slope of the line AP} = \frac{y - y_1}{x - x_1},$$

$$m_2 = \text{slope of the line BP} = \frac{y - y_2}{x - x_2},$$

Now, since the angle subtended at the point P in the semicircle APB is a right angle.

$$\therefore \quad m_1 m_2 = -1 \Rightarrow \frac{y - y_1}{x - x_1} \times \frac{y - y_2}{x - x_2} = -1$$

$$\Rightarrow \quad (y - y_1)(y - y_2) = -(x - x_1)(x - x_2).$$

$$\Rightarrow \quad (x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$$

This is the required equation of the circle having (x_1, y_1) and (x_2, y_2) as the coordinates of the end points of a diameter.

Remark:

If the coordinates of the end points of a diameter of a circle are given, we can also find the equation of the circle by finding the coordinates of the centre and radius. The centre is the mid point of the diameter and radius is half of the length of the diameter

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Illustration 11

If the abscissae and the ordinates of two points A and B be the roots of $ax^2 + bx + c = 0$ and $a'y^2 + b'y + c' = 0$ respectively, show that the equation of the circle described on AB as diameter is $aa'(x^2 + y^2) + a'bx + ab'y + (ca' + c'a) = 0$.

Let (x_1, y_1) and (x_2, y_2) be the coordinates of points A and B respectively.

It is given that x_1, x_2 are roots of $ax^2 + bx + c = 0$ ans y_1, y_2 are roots of $a'y^2 + b'y + c' = 0$.

$$\therefore \quad x_1 + x_2 = -\frac{b}{a},\ x_1 x_2 = \frac{c}{a},\ y_1 y_2 = -\frac{b'}{a'} \text{ and } y_1 y_2 = -\frac{c'}{a'}$$

The equation of the circle with AB as diameter is

$$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$$

$$\Rightarrow \quad x^2 + y^2 - x(x_1 + x_2) - y(y_1 + y_2) + x_1x_2 + y_1y_2 = 0.$$

$$\Rightarrow \quad x^2 + y^2 - x\left(-\frac{b}{a}\right) - y\left(-\frac{b'}{a'}\right) + \frac{c}{a} + \frac{c'}{a'} = 0$$

$$\Rightarrow \quad aa'(x^2 + y^2) + a'bx + ab'y + (ca' + c'a) = 0.$$

Illustration 12

If $y = 2x$ is a chord of the circle $x^2 + y^2 - 10x = 0$, find the equation of a circle with this chord as diameter.

The points of intersection of the given chord and the given circle are obtained by solving $y = 2x$ and $x^2 + y^2 - 10x = 0$ simultaneously.

Putting $y = 2x$ in $x^2 + y^2 - 10x = 0$, we get

$$5x^2 - 10x = 0 \Rightarrow 5x(x - 2) = 0 \Rightarrow x = 0, 2.$$

Putting $x = 0$ and $x = 2$ respectively in $y = 2x$, we get $y = 0$ and $y = 4$.

Thus, the coordinates of the points of intersection of the given line and the given circle are A (0, 0) and (2, 4).

The equation of the circle with chord AB as diameter is

$$(x - 0)(x - 2) + (y - 0)(y - 4) = 0 \Rightarrow x^2 + y^2 - 2x - 4y = 0.$$

Illustration 13

On the line joining (1, 0) and (3, 0) an equilateral triangle is drawn, having all its vertices in the first quadrant. Find the equations to the circles described on its sides as diameter.

Let (1, 0) and (3, 0) be the coordinates of the points A and B respectively. Then

$$AB = \sqrt{(1-3)^2 + (0-0)^2} = 2.$$

Let C (x_1, y_1) be the third vertex of the equilateral triangle ABC

Then, AB = BC = 2

Now,

$$AC = \sqrt{(x_1 - 1)^2 + (y_1 - 0)^2},\ BC = \sqrt{(x_1 - 3)^2 + (y_1 - 0)^2}$$

$$\therefore \quad AC = BC \Rightarrow AC^2 = BC^2 \Rightarrow (x_1 - 1)^2 + y_1^2 = (x_1 - 3)^2 + y_1^2$$

$$\Rightarrow \quad 4x_1 = 8 \Rightarrow x_1 = 2.$$

Again

$$AC = 2 \Rightarrow \sqrt{(x_1 - 1)^2 + y_1^2} = 2 \Rightarrow (x_1 - 1)^2 + y_1^2 = 4$$

$$\Rightarrow \quad (2 - 1)^2 + y_1^2 = 4 \qquad [\because x_1 = 2]$$

$$\Rightarrow \quad y_1 = \pm\sqrt{3}. \Rightarrow y_1 = \sqrt{3} \qquad [\because C(x_1, y_1) \text{ lies in first quadrant}]$$

So, the coordinates of C are $(2, \sqrt{3})$.

Now, the equation of the circle on AC as diameter is

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$(x-1)(x-2)+(y-0)(y-\sqrt{3})=0$

$\Rightarrow \quad x^2+y^2-3x-\sqrt{3}y+2=0.$

Similarly, the equations of circles with AB and BC as diameters are

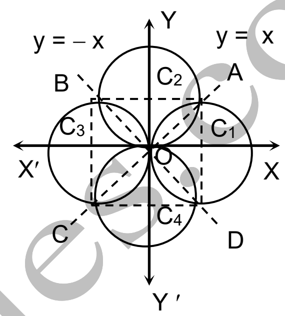
$(x-1)(x-3)+(y-0)(y-0)=0 \quad \Rightarrow x^2+y^2-4x+3=0$

and $(x-3)(x-2)+(y-0)(y-\sqrt{3})=0 \Rightarrow x^2+y^2-5x-\sqrt{3}y+6=0$ respectively.

Illustration 14

Find the equations to the circles which pass through the origin and cut off equal chords of length 'a' from the straight lines y = x and y = – x.

From Fig. there will be four such circles which pass through the origin and cut off equal chords of length a from the straight lines $y=\pm x$.



Since $\angle AOB=\angle BOC=\angle COD=\angle DOA=\pi/2$. Therefore, AB, BC, CD and DA are diameters of the four circles.

Now, $\angle XOA=\pi/4$ and OA = a

$\therefore \quad C_1A = a\sin\frac{\pi}{4}=\frac{a}{\sqrt{2}}$ and $OC_1=a\cos\frac{\pi}{4}=\frac{a}{\sqrt{2}}$

So, the coordinates of A are $\left(\frac{a}{\sqrt{2}},\frac{a}{\sqrt{2}}\right)$.

Similarly, the coordinates of B, C and D are

$\left(-\frac{a}{\sqrt{2}},\frac{a}{\sqrt{2}}\right),\left(-\frac{a}{\sqrt{2}},-\frac{a}{\sqrt{2}}\right)$ and $\left(\frac{a}{\sqrt{2}},-\frac{a}{\sqrt{2}}\right)$ respectively.

Now, the equation of the circle with AD as diameter is

$$\left(x-\frac{a}{\sqrt{2}}\right)\left(x-\frac{a}{\sqrt{2}}\right)+\left(y-\frac{a}{\sqrt{2}}\right)\left(y+\frac{a}{\sqrt{2}}\right)=0 \Rightarrow x^2+y^2-\sqrt{2}\,ax=0$$

Similarly, the equations of the required circles with BC, CD and AB as diameters are respectively.

$$\left(x+\frac{a}{\sqrt{2}}\right)\left(x+\frac{a}{\sqrt{2}}\right)+\left(y-\frac{a}{\sqrt{2}}\right)\left(y+\frac{a}{\sqrt{2}}\right)=0 \Rightarrow x^2+y^2+\sqrt{2}\,ax=0$$

$$\left(x+\frac{a}{\sqrt{2}}\right)\left(x-\frac{a}{\sqrt{2}}\right)+\left(y+\frac{a}{\sqrt{2}}\right)\left(y+\frac{a}{\sqrt{2}}\right)=0 \Rightarrow x^2+y^2+\sqrt{2}\,ay=0$$

$$\left(x-\frac{a}{\sqrt{2}}\right)\left(x+\frac{a}{\sqrt{2}}\right)+\left(y-\frac{a}{\sqrt{2}}\right)\left(y-\frac{a}{\sqrt{2}}\right)=0 \Rightarrow x^2+y^2-\sqrt{2}\,ay=0$$

Intercepts on the axes

The lengths of intercepts made by the circle $x^2+y^2+2gx+2fy+c=0$ with x and y axes are $2\sqrt{g^2-c}$ and $2\sqrt{f^2-c}$ respectively.

PROOF: Intercept on x-axis: Suppose the circle $x^2+y^2+2gx+2fy+c=0$ meets the x-axis at two points A_1 and A_2. Since on x-axis, y = 0. Therefore x-coordinates (abscissae) of points A_1 and A_2 are roots of the equation

$x^2+2gx+c=0 \quad$ ......(i)

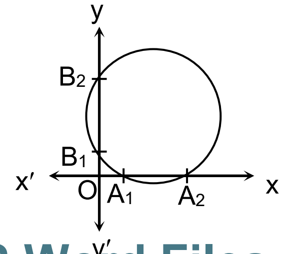
Let x_1 and x_2 be the abscissae of A_1 and A_2 respectively. Then x_1 and x_2 are roots of equation (i)

$\therefore \quad x_1+x_2=-2g$ and $x_1x_2=c$

Now, intercept on x-axis = $A_1A_2 = x_2-x_1$

$=\sqrt{(x_2+x_1)^2-4x_1x_2}=\sqrt{4g^2-4c}=2\sqrt{g^2-c}$

Similarly, intercept on y-axis = $2\sqrt{f^2-c}$.



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Remark:

(i) If $g^2 > c$, then the roots of equation $x^2 + 2gx + c = 0$ are real and distinct, so the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ meets the x-axis in two real and distinct points and the length of the intercept on x-axis is $2\sqrt{g^2 - c}$.

(ii) If $g^2 = c$, then the roots of the equation $x^2 + 2gx + c = 0$ are real and equal, so the circle touches x-axis and the intercept on x-axis is zero.

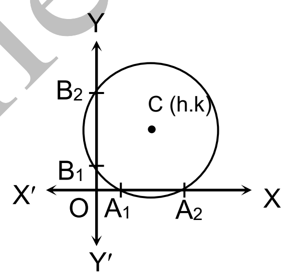
(iii) If $g^2 < c$, then the roots of the equation $x^2 + 2gx + c = 0$ are imaginary, so the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ does not meet x-axis in real points.

(iv) Similarly, the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ cuts the y-axis in real and distinct points, touches or does not meet in real points according as $f^2 >, =$ or $< c$.

Illustration 15

Two rods of length a and b slide along the coordinate axes, which are rectangular, in such a way that their ends are always concyclic; prove that the locus of the centre of the circle passing through these ends is the curve $4(x^2 - y^2) = a^2 - b^2$.

Let $A_1 A_2$ and $B_1 B_2$ be two rods of length a and b which slide along OX and OY respectively. Let $x^2 + y^2 + 2gx + 2fy + c = 0$ be the circle passing through A_1, A_2, B_1, B_2. Then



$|A_1A_2|$ = Intercept on x-axis

$= 2\sqrt{g^2 - c} \Rightarrow a = 2\sqrt{g^2 - c}$ .....(i)

And $|B_1B_2|$ = Intercept on y-axis

$= 2\sqrt{f^2 - c} \Rightarrow b = 2\sqrt{f^2 - c}$ .....(ii)

Here c is the variable which is to be eliminated to find the locus of the centre $(-g,-f)$ of the circle $x^2 + y^2 + 2gx + 2fy + c = 0$.

From (i) and (ii), we get $a^2 - b^2 = 4(g^2 - f^2)$. Therefore,

locus of $(-g, -f)$ is $a^2 - b^2 = 4(x^2 - y^2)$ [Replacing $-g$ by x and $-f$ by y]

Hence, the locus of the centre of the circle is

$a^2 - b^2 = 4(x^2 - y^2)$.

Position of a point with respect to a circle

A point (x_1, y_1) lies outside, on or inside a circle $S = x^2 + y^2 + 2gx + 2fy + c = 0$ according as $S_1 > =$ or < 0, where $S_1 = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$.

PROOF: Let P $(x_1\ y_1)$ be the given point and let C be the centre of the circle. Then the coordinates of C are $(-g, -f)$.

Now, $CP = \sqrt{(x_1 + g)^2 + (y_1 + f)^2}$.

The point P lies outside, on or inside the circle according as

CP > = or < the radius of the circle

i.e., $\sqrt{(x_1 + g)^2 + (y_1 + f)^2} \ > = \text{or} < \sqrt{g^2 + f^2 - c}$ (radius of the circle)

$\Rightarrow (x_1 + g)^2 + (y_1 + f)^2 > = \text{or} < g^2 + f^2 - c$

$\Rightarrow x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c >= \text{or} < 0$

or $S_1 > =$ or < 0, where $S_1 = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$.

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Illustration 16

Discuss the position of the points (1, 2) and (6, 0) with respect to the circle $x^2 + y^2 - 4x + 2y - 11 = 0$

We have $x^2 + y^2 - 4x + 2y - 11 = 0$ or S = 0, where $S = x^2 + y^2 - 4x + 2y - 11$.

For the point (1, 2), we have

$$S_1 = 1^2 + 2^2 - 4 \times 1 + 2 \times 2 - 11 < 0.$$

For the point (6, 0), we have

$$S^2 = 6^2 + 0^2 - 4 \times 6 + 2 \times 0 - 11 > 0.$$

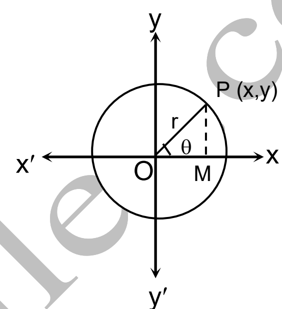
Hence, the point (1, 2) lies inside the circle and the point (6, 0) lies outside the circle.

Equations of a circle in parametric form

Parametric equations of $x^2 + y^2 = r^2$:

Let P (x, y) be any point on the circle $x^2 + y^2 = r^2$, and let OP make an angle θ with the positive direction of x-axis. Let PM be perpendicular from P on x-axis.

Then $\angle MOP = \theta$.

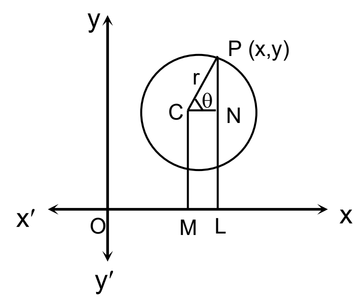


$\therefore$ $x = OM = r\cos\theta$ and $y = PM = r\sin\theta$.

Thus, $x = r\cos\theta$, $y = r\sin\theta$ are the required parametric from of the equation $x^2 + y^2 = r^2$, where θ is a parameter such that $0 \le \theta < 2\pi$.

Parametric equations of $(x - a)^2 + (y - b)^2 = r^2$:

Let P(x, y) be any point on the circle $(x - a)^2 + (y - b)^2 = r^2$. Let PL and CM be perpendiculars from P and C respectively on x-axis and let CN be perpendicular from C on PL.



Let $\angle NCP = \theta$. Then, $x = OL = OM + ML$

$= OM + CN$

$= a + r\cos\theta$ $\quad$ [$\because$ $CN = CP\cos\theta = r\cos\theta$]

And, $y = PL = PN + NL = CM + PN$

$= b + r\sin\theta$ $\quad$ [$\because$ $PN = CP\sin\theta = r\sin\theta$]

Hence, $x = a + r\cos\theta$; $y = b + r\sin\theta$ are the required parametric equations of the circle $(x - a)^2 + (y - b)^2 = r^2$, where $0 \le \theta < 2\pi$.

Illustration 17

Write the parametric equations of the circle $x^2 + y^2 - 2x - 4y - 4 = 0$.

We have : $x^2 + y^2 - 2x - 4y - 4 = 0$

$\Rightarrow$ $(x^2 - 2x + 1) + (y^2 - 4y + 4) = 9$

$\Rightarrow$ $(x - 1)^2 + (y - 2)^2 = 3^2$

Hence, the parametric equations of the given circle are $x = 1 + 3\cos\theta$, $y = 2 + 3\sin\theta$, where θ is a variable lying between 0 and 2π.

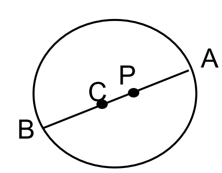
Maximum and Minimum distance of a point from a circle

Case I Point $P(x_1, y_1)$ lies inside the circle $x^2 + y^2 + 2gx + 2fy + c = 0$

Minimum distance of P from the circle

$= PA = CA - CP = |r - CP|$

Maximum distance of P from the circle



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= PB = CB + CP = |r + CP|

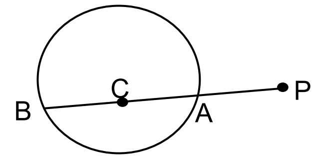
Case II Point P(x_1, y_1) lies outside the circle $x^2 + y^2 + 2gx + 2fy + c = 0$

Minimum distance of P from the circle

= PA = CP – CA = |CP – r|

Maximum distance of P from the circle

= PB = CP + CB = |r + CP|



Case III Point P (x_1, y_1) lies on the circle $x^2 + y^2 + 2gx + 2fy + c = 0$

Minimum distance of P from the circle = 0

Maximum distance of P from the circle = PA = 2r

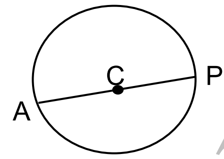


Illustration 18

Find the largest and smallest distance of the point (2, – 7) from the circle $x^2 + y^2 - 14x - 10y - 151 = 0$

$S_1 = -56 < 0$

$\therefore$ Point P (2, – 7) lies inside the circle

Centre $\equiv$ C (7, 5) radius r = 15, CP = $\sqrt{(7-2)^2 + (5+7)^2} = 13$

$\therefore$ Largest distance = |r + CP| = 15 + 13 = 28

Smallest distance = |r – CP| = 15 – 13 = 2

Practice Assignment - I

1. In each of the following find the equation of the circle

(i) centre (0, 2) and radius 2

(ii) centre (–2, 3) and radius 4

(iii) centre $\left(\frac{1}{2}, \frac{1}{4}\right)$ and radius $\frac{1}{12}$

(iv) centre (1, 1) and radius $\sqrt{2}$

(v) centre (–a, –b) and radius $\sqrt{a^2 - b^2}$.

2. In each of the following find the centre and radius of the circle

(i) $(x + 5)^2 + (y - 3)^2 = 36$

(ii) $x^2 + y^2 - 4x - 8y - 45 = 0$

(iii) $x^2 + y^2 - 8x + 10y - 12 = 0$

(iv) $2x^2 + 2y^2 - x = 0$

(v) $ax^2 + (2a - 3)y^2 - 4x - 1 = 0$

3. Prove that the radii of the circles

$x^2 + y^2 = 1$, $x^2 + y^2 - 2x - 6y = 6$ and $x^2 + y^2 - 4x - 12y = 9$ are in A.P.

4. If $2x^2 + \lambda xy + (\lambda + 2)y^2 + (\lambda - 4)x + 6y - 5 = 0$ is the equation of a circle then , find its radius.

5. If $2(x^2 + y^2) + 4\lambda x + \lambda^2 = 0$ represents a circle of meaningful radius then, find the range of real values of λ.

6. Find the equation of the circle concentric with the circle $x^2 + y^2 - 8x + 6y - 5 = 0$ and passing through the point (–2, –7).

7. Find the area of an equilateral triangle inscribed in the circle $x^2 + y^2 + 2gx + 2fy + c = 0$.

8. Find the equation of the circle whose centre is (1, 2) and which passes through the point (4, 6).

9. If the equations of the two diameters of a circle are x + y = 6 and x + 2y = 4 and the radius of the circle is 10, find the equation of the circle.

10. Find the equation of the circle which touches the axes and whose centre lies on the line x – 2y = 3.

11. The abscissae of two points A and B are the roots of the equation $x^2 + 2ax - b^2 = 0$ and their ordinates are the roots of the equation $x^2 + 2px - q^2 = 0$. Find the equation of the circle with AB as diameter.

12. The sides of a square are x = 1, x = 3, y = 2 and y = 4. Find the equation of the circle drawn on the diagonals of the square as its diameter.

13. Find the equation of the circle which passes through the points (1, –2) and (4, –3) and whose centre lies on the line 3x + 4y = 7.

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14. Find the equation of circle passing through the points (2, 3), (–1, 1) and whose centre lies on the line $x - 3y - 11 = 0$.

15. Find the equation of circle with radius 5 whose centre lies on x-axis and passes through the point (2, 3).

16. Find the equation of the circle passing through the point (0, 0) and the points where the straight line $3x + 4y = 12$, meets the axes of co-ordinates.

17. Find the equation of the circle which passes through (3,–6) and touches both the axes. How many such circles can be drawn?

18. The circle $x^2+y^2-2x-2y+1= 0$ is rolled along the positive direction of x axis and makes one complete roll. Find its equation in new position.

19. Find the equation of the circle passing through the three non collinear points (1,1), (2,–1), (3,2).

20. Find the equation of the circle passing through (1, 0) and (0, 1) and having the smallest possible radius.

21. If A(–4, 3) and B(12,–1) be the extremities of a diameter of a circle, what length does the circle intercept on y–axis?

22. Find the equation to the circle touching the y-axis at a distance –3 from the origin and intercepting a length 8 on the x-axis.

23. Find the equation of the circle which passes through the origin and makes intercepts of length a and b on x axis and y axis respectively.

24. Does the point (–2.5, 3.5) lie inside, outside or on the circle $x^2 + y^2 = 25$?

25. Find whether the point $\left(1, \frac{\sqrt{11}+6}{6}\right)$ lies inside or outside the circle $3x^2 + 3y^2 - 5x - 6y + 4 = 0$.

26. Find the set of values of a for which the point (a–1, a+1) lies out side the circle $x^2 + y^2 = 8$ and inside the circle $x^2 + y^2 - 12x + 12y - 62 = 0$.

27. Find the parametric equations of the circle $x^2+y^2 +x + \sqrt{3}\,y = 0$.

28. Find the equation of a circle on which the co–ordinates of any point are $(2 + 4\cos\theta, -1 + 4\sin\theta)$, θ being the parameter.

29. Find the parametric form of the equation of the circle $x^2 + y^2 + px + py = 0$

30. Find the maximum and minimum distance of the point (10,7) from the circle $x^2 + y^2 - 4x - 2y - 20 = 0$.

Intersection of a straight line and a circle

Let the equation of the circle be

$$x^2 + y^2 = a^2 \qquad \text{...(i)}$$

and the equation of the line be

$$y = mx + c \qquad \text{....(ii)}$$

Since the points of intersection of (i) and (ii) satisfy each of them, therefore to find the points of intersection of (i) and (ii), we have to solve them simultaneously.

Substituting the value of y from (ii) in (i), we get

$$x^2 + (mx + c)^2 = a^2 \Rightarrow x^2 (1 + m^2) + 2mcx + (c^2 - a^2) = 0 \qquad \text{....(iii)}$$

This, being quadratic equation in x, gives two values of x which are the abscissae of the points of intersection of (i) and (ii). substituting the values of x in (ii), we get the corresponding values of y.

These corresponding values of x and y are the coordinates of the points of intersection of (i) and (ii).

Thus, a line may cut a circle in at most two points. The points of intersection are real and distinct, coincident or imaginary according as the roots of the quadratic equation (iii) are real and different, coincident or imaginary as discussed in the following cases.

CASE 1. When points of intersection are real and distinct: In case the equation (iii) has two distinct roots.

$$\Rightarrow \text{Discriminant} > 0 \Rightarrow 4m^2c^2 - 4(1 + m^2)(c^2 - a^2) > 0$$

$$\Rightarrow 4a^2 (1 + m^2) - 4c^2 > 0 \Rightarrow a^2(1 + m^2) > c^2 \Rightarrow a^2 > \frac{c^2}{1+m^2}$$

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$\Rightarrow a > \left|\frac{c}{\sqrt{1+m^2}}\right|$

$\Rightarrow$ a > (length of the perpendicular from the centre (0, 0) to y = mx + c)

Thus, a line intersects a given circle at two distinct points if the length of the perpendicular form the centre is less then the radius of the circle.

CASE 2. When the points of intersection are coincident: in this case the equation (iii) has two equal roots.

$\therefore$ Discriminant = 0 $\Rightarrow 4m^2c^2 - 4(1 + m^2)(c^2 - a^2) = 0$

$\Rightarrow c^2 = a^2(1 + m^2) \Rightarrow a = \left|\frac{c}{\sqrt{1+m^2}}\right|$

$\Rightarrow$ a = length of the perpendicular from the centre (0, 0) to y = mx + c.

Thus, a line touches a circle if the length of the perpendicular from the centre is equal to the radius of the circle.

CASE 3. When the points of intersection are imaginary: in this case equation (iii) has imaginary roots.

$\therefore$ Discriminant < 0 $\Rightarrow 4m^2c^2 - 4(1 + m^2)(c^2 - a^2) < 0$

$\Rightarrow a^2(1 + m^2) - c^2 < 0 \Rightarrow a^2(1 + m^2) < c^2$

$\Rightarrow a < \left|\frac{c}{\sqrt{1+m^2}}\right|$

$\Rightarrow$ a < length of the perpendicular from the centre (0, 0) to y = mx + c.

Thus, a line does not intersects a circle if the length of the perpendicular from the centre is greater than the radius of the circle.

Illustration 19

Find the points of intersection of the line 2x + 3y = 18 and the circle $x^2 + y^2 = 25$.

We have :

$2x + 3y = 18$ ...(i)

and $x^2 + y^2 = 25$ ....(ii)

Substituting $y = \frac{18-2x}{3}$ obtained from (i) in (ii), we get

$x^2 + \left(\frac{18-2x}{3}\right)^2 = 25 \Rightarrow 13x^2 - 72x + 99 = 0$

$\Rightarrow (x - 3)(13x - 33) = 0 \Rightarrow x = 3$ or $x = \frac{33}{13}$.

Substituting the values of x in (i), we get y = 4 or $y = \frac{56}{13}$ respectively.

Hence, the points of intersection of the given line and the given circle are (3, 4) and $\left(\frac{33}{13}, \frac{56}{13}\right)$.

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The length of the intercept cut from a line by a circle

The length of the intercept cut from the line y = mx + c by the circle $x^2 + y^2 = a^2$ is $2\sqrt{\dfrac{a^2(1+m^2)-c^2}{(1+m^2)}}$.

PROOF: Let P (x_1, y_1) and Q (x_2, y_2) be the points intersection of the circle and the line whose equations are

$x^2 + y^2 = a^2$ ....(i)

and $y = mx + c$ ....(ii)

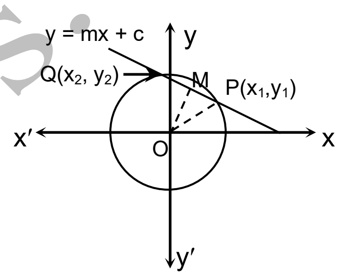
Substituting the value of y from (ii) in (i), we have

$x^2 + (mx + c)^2 = a^2 \Rightarrow x^2(1 + m^2) + 2\,mcx + (c^2 - a^2) = 0$

This is a quadratic equation in x. clearly, x_1, x_2 are the roots of this equation.

$\therefore x_1 + x_2 = \dfrac{-2mc}{1+m^2}$ and $x_1x_2 = \dfrac{c^2-a^2}{1+m^2}$

$\therefore x_2 - x_1 = \sqrt{(x_1+x_2)^2 - 4x_1x_2} = \sqrt{\dfrac{4m^2c^2}{(1+m^2)^2} - \dfrac{4(c^2-a^2)}{1+m^2}}$



$= \dfrac{2}{1+m^2}\sqrt{a^2(1+m^2)-c^2}$ ....(iii)

Since P (x_1, y_1) and Q (x_2, y_2) lie on y = mx + c.

$\therefore y_1 = mx_1 + c$ and $y_2 = mx_2 + c \Rightarrow y_2 - y_1 = m(x_2 - x_1)$

Now,

$PQ = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2}$

$= \sqrt{(x_2-x_1)^2 + m^2(x_2-x_1)^2}$ [$\because y_2 - y_1 = m(x_2 - x_1)$]

$= (x_2 - x_1)\sqrt{1+m^2}$

$= \dfrac{2}{1+m^2}\sqrt{a^2(1+m^2)-c^2}\,.\sqrt{1+m^2}$ [Using (iii)]

$= 2\sqrt{\dfrac{a^2(1+m^2)-c^2}{1+m^2}}$.

ALITER Draw OM ⊥ PQ. join OP.

Now, OM = (Length of the ⊥ from O (0, 0) to y = mx + c)

$= \dfrac{c}{\sqrt{1+m^2}}$ and OP = a

$PQ = 2PM = 2\sqrt{OP^2 - OM^2}$

$= 2\sqrt{a^2 - \dfrac{c^2}{1+m^2}} = 2\sqrt{\dfrac{a^2(1+m^2)-c^2}{1+m^2}}$

Condition of tangency

The line y = mx + c touches the circle $x^2 + y^2 = a^2$ if the length of the intercept is zero.

i.e., $PQ = 0 \Rightarrow 2\sqrt{\dfrac{a^2(1+m^2)-c^2}{1+m^2}} = 0 \Rightarrow a^2(1+m^2) - c^2 = 0$.

$\Rightarrow$ $c = \pm a\sqrt{1+m^2}$ which is the required condition.

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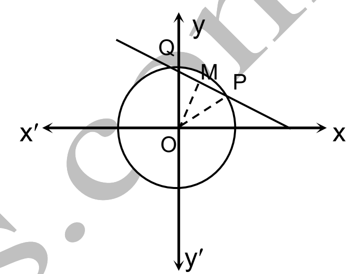
REMARK **The method used above is a general method and may be applied to any curve. In case of a circle we can also use the result that if a line touches a circle, then the length of the perpendicular from the centre upon the line is equal to the radius of the circle. Thus, the condition of tangency can also be obtained by using this result.**

Illustration 20

Find the length of the intercept on the straight line $\frac{x}{a}+\frac{y}{b}=1$ by the circle $x^2+y^2=r^2$



Let P and Q be the points of intersection of the line $\frac{x}{a}+\frac{y}{b}=1$ and the circle $x^2+y^2=r^2$.

Let OM be the perpendicular from O on the line $\frac{x}{a}+\frac{y}{b}=1$

Then, $OM=\left|\frac{\frac{0}{a}+\frac{0}{b}-1}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}}}\right|=\frac{1}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}}}=\frac{ab}{\sqrt{a^2+b^2}}$

$\therefore$ $PQ=2\,PM=2\sqrt{OP^2-OM^2}=2\sqrt{r^2-\frac{a^2b^2}{a^2+b^2}}$

Illustration 21

For what value of c will the line $y=2x+c$ be a tangent to the circle $x^2+y^2=5$?

We have : $y=2x+c$ or $2x-y-c=0$ ....(i)

And $x^2+y^2=5$ ....(ii)

If the line (i) touches the circle (ii) then,

Length of the $\perp$ from the centre (0, 0) = radius of circle (ii)

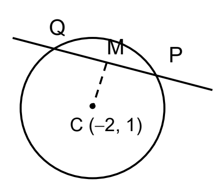
$\Rightarrow$ $\left|\frac{2\times 0-0+c}{\sqrt{2^2+(-1)^2}}\right|=\sqrt{5}\Rightarrow\left|\frac{c}{\sqrt{5}}\right|=\pm\sqrt{5}\Rightarrow c=\pm 5$.

Hence, the line (i) touches the circle (ii) for $c=\pm 5$.

Illustration 22

Find the coordinates of the middle point of the chord which the circle $x^2+y^2+4x-2y-3=0$ cuts off the line $x-y+2=0$.

The coordinates of the centre C of the given circle are (– 2, 1). Let the line AB whose equation is $x-y+2=0$ intersect the circle at P and Q. let M be the mid-point of PQ. Then M is the point of intersection of PQ and a line passing through C and perpendicular to PQ. The equation of a line passing through C(–2, 1) and perpendicular to PQ is



$y-1=-1\,(x+2)$ or $x+y+1=0$

Solving $x-y+2=0$ and $x+y+1=0$, we obtain $x=-\frac{3}{2}$, $y=\frac{1}{2}$.

Hence, the coordinates of the mid-point of PQ are $\left(-\frac{3}{2},\frac{1}{2}\right)$.

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Illustration 23

If $\ell x + my = 1$ touches the circle $x^2 + y^2 = a^2$, prove that the point ℓ, m lies on the circle $x^2 + y^2 = a^{-2}$.

The line $\ell x + my = 1$ touches the circle $x^2 + y^2 = a^2$, if length of the perpendicular from the centre (0, 0) to $\ell x + my = 1$ is equal to the radius of the circle $x^2 + y^2 = a^2$

$$\Rightarrow \quad \left|\frac{\ell.0 + m.0 - 1}{\sqrt{\ell^2 + m^2}}\right| = a \Rightarrow \ell^2 + m^2 = \frac{1}{a^2}$$

$\Rightarrow$ (ℓ, m) satisfies equation $x^2 + y^2 = a^{-2}$

$\Rightarrow$ (ℓ, m) lies on the circle $x^2 + y^2 = a^{-2}$

Different forms of the equations of Tangents

SLOPE FORM.

The equation of a tangent of slope m to the circle $x^2 + y^2 = a^2$ is $y = mx \pm a\sqrt{1+m^2}$. The coordinate of the point of contact are $\left(\pm\frac{am}{\sqrt{1+m^2}}, \pm\frac{a}{\sqrt{1+m^2}}\right)$.

PROOF: Let m be the slope of the tangent to the circle $x^2 + y^2 = a^2$. Then the equation of the tangent is of the form $y = mx + c$, where c is to be calculated from the fact that the line $y = mx + c$ is a tangent to $x^2 + y^2 = a^2$.

Since $y = mx + c$ is a tangent to $x^2 + y^2 = a^2$

$\therefore$ length of the perpendicular from the centre (0, 0) = radius

$$\Rightarrow \left|\frac{m \times 0 - 0 + c}{\sqrt{m^2 + (-1)^2}}\right| = a \Rightarrow c = \pm a\sqrt{1+m^2}.$$

Substituting this value of c in $y = mx + c$, we get $y = mx \pm a\sqrt{1+m^2}$ which are the required equations of the tangents.

Point of contact solving

$x^2 + y^2 = a^2$ and $y = mx \pm a\sqrt{1+m^2}$

simultaneously, we get

$x = \pm\frac{am}{\sqrt{1+m^2}}$ and $y = \mp\frac{a}{\sqrt{1+m^2}}$

Thus, the coordinates of the points of contact are

$$\left(\pm\frac{am}{\sqrt{1+m^2}}, \mp\frac{a}{\sqrt{1+m^2}}\right)$$

Thus, the equation of tangents of slope m to the circle $x^2 + y^2 = a^2$ are $y = mx \pm a\sqrt{1+m^2}$ and the coordinates of the points of contact are $\left(\pm\frac{am}{\sqrt{1+m^2}}, \mp\frac{a}{\sqrt{1+m^2}}\right)$

POINT FORM The equation of the tangent at the point P (x_1, y_1) to a circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is

$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$

PROOF: let $P(x_1, y_1)$ be a given point on the circle

$x^2 + y^2 + 2gx + 2fy + c = 0$ ....(i)

Let Q (x_2, y_2) be any other point on this circle.

Since points $P(x_1, y_1)$ and $Q(x_2, y_2)$ lie on the circle. Therefore

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$$x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c = 0 \quad \text{....(ii)}$$

and $$x_2^2 + y_2^2 + 2gx_2 + 2fy_2 + c = 0 \quad \text{.....(iii)}$$

Subtracting (iii) from (ii), we have

$$(x_2^2 - x_1^2) + (y_2^2 - y_1^2) + 2g(x_2 - x_1) + 2f(y_2 - y_1) = 0$$

$\Rightarrow$ $(x_2 - x_1)(x_2 + x_1 + 2g) + (y_2 - y_1)(y_2 + y_1 + 2f) = 0$

$\Rightarrow$ $$\frac{y_2 - y_1}{x_2 - x_1} = -\left(\frac{x_2 + x_1 + 2g}{y_2 + y_1 + 2f}\right) \quad \text{.....(iv)}$$

Now, the equation of the chord PQ is $y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$

or $$y - y_1 = -\left(\frac{x_2 + x_1 + 2g}{y_2 + y_1 + 2f}\right)(x - x_1) \quad \text{.....(v) (using iv)}$$

Now, when $Q \to P$, then $x_2 \to x_1$, $y_2 \to y_1$ and the chord PQ becomes the tangent at P.
So the equation of the tangent at P is

$$y - y_1 = -\frac{2x_1 + 2g}{2y_1 + 2f}(x - x_1)$$

$\Rightarrow$ $(x - x_1)(x_1 + g) + (y - y_1)(y_1 + f) = 0$

$\Rightarrow$ $xx_1 + yy_1 + gx + fy = x_1^2 + y_1^2 + gx_1 + fy_1.$

$\Rightarrow$ $xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c.$

$\Rightarrow$ $xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$ (Using (ii))

This is required equation of the tangent at (x_1, y_1) to the circle
$x^2 + y^2 + 2gx + 2fy + c = 0$

Algorithm to write the equation of tangent to a circle at a given point

To write the equation of the tangent to a circle at a given point (x_1, y_1) we proceed as follows:

STEP1. In the equation of the circle substitute xx_1 for x^2, yy_1 for y^2, $\frac{x + x_1}{2}$ for x, $\frac{y + y_1}{2}$ for y and keep the constant as such.

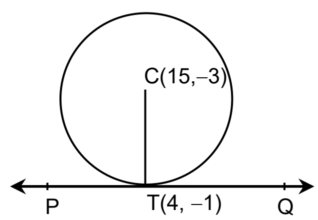
STEP2. Simplify the equation obtained in step I
The equation so obtained is the desired equation of the tangent.

Illustration 24

Find the equation of the tangent to the circle $x^2 + y^2 - 30x + 6y + 109 = 0$ at (4, –1).

$x^2 + y^2 - 30x + 6y + 109 = 0$ .....(i)

Centre of the circle (i) is C(15, –3)



From Fig. Slope of line CT is = $\frac{2}{-11}$ $\therefore$ Slope of the line PT = $\frac{11}{2}$

$\therefore$ Equation of tangent in point slope form is

$y + 1 = \frac{11}{2}(x - 4)$ $\Rightarrow$ $11x - 2y = 46.$

Alternative method

Replacing x^2 by 4x, y^2 by $(-1)y = -y$, x by $\frac{x+4}{2}$ and y by $\frac{y + (-1)}{2}$ in the equation of the circle, we obtain

$$4x + (-y) - 30\left(\frac{x+4}{2}\right) + 6\left(\frac{y + (-1)}{2}\right) + 109 = 0$$

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$\Rightarrow$ $4x - y - 5x - 60 + 3y - 3 + 109 = 0$ $\Rightarrow 11x - 2y - 46 = 0$

Hence, the required equation of the tangent is $11x - 2y - 46 = 0$

Illustration 25

Prove that the tangents to the circle $x^2 + y^2 = 169$ at (5, 12) and (12, – 5) are perpendicular to each other.

The equations of tangent to $x^2 + y^2 = 169$ at (5, 12) and (12, – 5) are

$5x + 12y = 169$ ....(1) $12x - 5y = 169$ ...(2)

respectively.

Now. m_1 = slope of (1) = – 5/12, m_2 = slope of (2) = 12/ 5.

Clearly, $m_1m_2 = -1$.

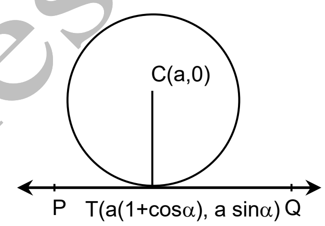
Hence, (1) and (2) are perpendicular to each other.

Illustration 26

Find the equation of the tangent to the circle $x^2 + y^2 - 2ax = 0$ at the point $(a(1 + \cos\alpha), a \sin\alpha)$

$x^2 + y^2 - 2ax = 0$ ....(i)

Centre of circle (i) (a, 0)



Slope of CT is = $\dfrac{a\sin\alpha - 0}{a(1+\cos\alpha) - a} = \dfrac{\sin\alpha}{\cos\alpha}$

$\therefore$ Slope of PT is = $\dfrac{-\cos\alpha}{\sin\alpha}$

$\therefore$ Equation of PT

$y - a\sin\alpha = \dfrac{-\cos\alpha}{\sin\alpha}(x - a(1 + \cos\alpha))$

$\Rightarrow$ $x\cos\alpha + y\sin\alpha - a(1 + \cos\alpha) = 0$

Alternative method

Replacing x^2 by $x(a(1 + \cos\alpha)$, y^2 by $y(a\sin\alpha)$ and x by $\dfrac{x + a(1+\cos\alpha)}{2}$ in the equation of the circle, we get

$x \cdot a(1 + \cos\alpha) + y\, a\sin\alpha - 2a\left\{\dfrac{x + a(1+\cos\alpha)}{2}\right\} = 0$

$\Rightarrow$ $x(1 + \cos\alpha) + y\sin\alpha - \{x + a(1 + \cos\alpha)\} = 0$

$x\cos\alpha + y\cos\alpha - a(1 + \cos\alpha) = 0$

This is the equation of the required tangent.

Illustration 27

Prove that the line $lx + my + n = 0$ touches the circle $(x - a)^2 + (y - b)^2 = r^2$ if $(al + bm + n)^2 = r^2(l^2 + m^2)$

if the line $lx + my + n = 0$ touches the circle $(x - a)^2 + (y - b)^2 = r^2$, then Length of the perpendicular from the centre = radius

$\Rightarrow$ [length of the $\perp$ from (a, b) to the line $lx + my + n = 0$] = r

$\Rightarrow$ $\left|\dfrac{la + mb + n}{\sqrt{l^2 + m^2}}\right| = r$ $\Rightarrow (la + mb + n)^2 = r^2(l^2 + m^2)$.

Illustration 28

Find the value of c so that $y = mx + c$ may be a tangent to the circle $x^2 + y^2 = 4y$ for all values of m.

We have : $y = mx + c$ or $mx - y + c = 0$ ...(i)

and $x^2 + y^2 = 4y$ or $x^2 + y^2 - 4y = 0$ ...(ii)

The coordinates of centres and radius of (ii) are (0, 2) and 2 respectively.

If the line (i) touches (ii) then

Length of the $\perp$ from (0, 2) to (i) = radius of (ii)

$$\Rightarrow \quad \left|\frac{m\times 0-2+c}{\sqrt{m^2+(-1)^2}}\right| = 2 \Rightarrow \left|\frac{-2+c}{\sqrt{m^2+1}}\right| = 2 \Rightarrow \frac{-2+c}{\sqrt{m^2+1}} = \pm 2$$

$$\Rightarrow \quad c-2 = \pm 2\sqrt{1+m^2} \;\Rightarrow\; c = 2\pm 2\sqrt{1+m^2}$$

Illustration 29

Find the equation of the tangents to the circle $x^2 + y^2 = 4$ which are perpendicular to the line $12x - 5y + 9 = 0$. Also, find the point of contact.

The slope of a line perpendicular to the line $12x - 5y + 9 = 0$ is $-5/12$. We have to find the equations of tangents of slope $m = -5/12$. So, the equations of the tangents are

$$y = -\frac{5}{12}x \pm 2.\sqrt{1+\left(-\frac{5}{12}\right)^2} \;\Rightarrow\; 5x + 12y = \pm 26 \text{ (using } y = mx \pm a\sqrt{1+m^2}\text{)}$$

The points of contact are $\left(\pm\frac{am}{\sqrt{1+m^2}}, \pm\frac{a}{\sqrt{1+m^2}}\right)$ i.e. $\left(\pm\frac{10}{13}, \pm\frac{24}{13}\right)$

Illustration 30

Find the equations of the tangents to the circle $x^2 + y^2 - 6x + 4y = 12$ which are parallel to the straight line $4x + 3y + 5 = 0$.

We have: $4x + 3y + 5 = 0$ ...(i)

and $x^2 + y^2 - 6x + 4y - 12 = 0$ ...(ii)

The coordinates of centre and radius of circle (ii) are (3, –2) and 5 respectively.

The equation of a line parallel to $4x + 3y + 5 = 0$ is

$4x + 3y + \lambda = 0$ ...(iii)

This line will touch the circle if length of the $\perp$ from the centre = radius

$$\Rightarrow \quad \left|\frac{4\times 3+3\times -2+\lambda}{\sqrt{4^2+3^2}}\right| = 5$$

$$\Rightarrow \quad \left|\frac{\lambda+6}{5}\right| = 5 \Rightarrow \frac{\lambda+6}{5} = \pm 5 \Rightarrow \lambda + 6 = \pm 25$$

$\Rightarrow \quad \lambda = 19, \lambda = -31$

Substituting these values of λ in (iii), we obtain

$4x + 3y + 19 = 0$ and $4x + 3y - 31 = 0$

These are the required tangents.

Illustration 31

Prove that the line $x + y = 5$ touches the circle $x^2 + y^2 - 2x - 4y + 3 = 0$. Find the point of contact.

The equation of the tangent to the given circle at (x_1, y_1) is

$xx_1 + yy_1 - (x + x_1) - 2(y + y_1) + 3 = 0$

$\Rightarrow \quad (x_1 - 1)x + (y_1 - 2)y = x_1 + 2y_1 - 3$ ...(i)

The equation of the given line is $x + y = 5$ ...(ii)

Let this line be tangent to the given circle at (x_1, y_1). Then (i) and (ii) represent the same line.

$$\therefore \quad \frac{x_1-1}{1} = \frac{y_1-2}{1} = \frac{x_1+2y_1-3}{5} = \lambda \text{ (say)}$$

$\Rightarrow \quad x_1 = \lambda + 1,\; y_1 = \lambda + 2$ and $x_1 + 2y_1 - 3 = 5\lambda$

$\Rightarrow \quad (\lambda + 1) + 2(\lambda + 2) - 3 = 5\lambda \Rightarrow \lambda = 1.$

Putting the value of λ in $x_1 = \lambda + 1$ and $y_1 = \lambda + 2$, we get

$\therefore \quad x_1 = 2$ and $y_1 = 3$.

Hence, the line x + y = 5 touches the given circle at (2, 3).

Illustration 32

Determine the radius of the circle, two of whose tangents are the lines $2x + 3y - 9 = 0$ and $4x + 6y + 19 = 0$.

The given tangents are

$2x + 3y - 9 = 0$ ...(1)

and $4x + 6y + 19 = 0$

or $2x + 3y + \frac{19}{2} = 0$ ....(2)

which are parallel, then distance between parallel tangents must be diameter of the circle. Then diameter

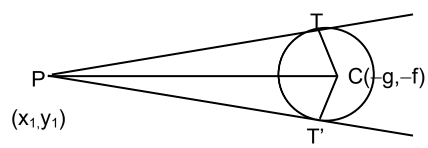
$$= \frac{\left|\frac{19}{2} - (-9)\right|}{\sqrt{2^2 + 3^2}} = \frac{37}{2\sqrt{13}}$$

$\Rightarrow$ **radius = $\frac{1}{2}$ diameter = $\frac{37}{4\sqrt{13}}$**

Length of the tangent from a point to a circle

The length of the tangent from the point P (x_1, y_1) to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is equal to $\sqrt{x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c}$.

PROOF. Let PT and PT' be two tangents from the given point P (x_1, y_1) to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$. The centre of the circle is C $(-g, -f)$ and radius = CT = $\sqrt{g^2 + f^2 - c}$.



Now, from the right angled triangle PTC, we have

$$CP^2 = CT^2 + PT^2$$

$\Rightarrow \quad PT^2 = CP^2 - CT^2 = (x_1 + g)^2 + (y_1 + f)^2 - (\sqrt{g^2 + f^2 - c})^2$

$\Rightarrow \quad PT^2 = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$

$\Rightarrow \quad PT = \sqrt{x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c}$

Remark.1. If PT is the length of the tangent from a point P to a given circle, then PT^2 is called the **power of the point** with respect to the given circle.

Remark.2. If we write $S = x^2 + y^2 + 2gx + 2fy + c$ and $S_1 = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$, then the equation of the circle is S = 0 and the length of the tangent from P (x_1, y_1) is $\sqrt{S_1}$. The power of the point P (x_1, y_1) is S_1.

Illustration 33

Find the length of the tangents drawn from the point (3, –4) to the circle $2x^2 + 2y^2 - 7x - 9y - 13 = 0$.

The equation of the given circle is

$2x^2 + 2y^2 - 7x - 9y - 13 = 0. \Rightarrow x^2 + y^2 - \frac{7}{2}x - \frac{9}{2}y - \frac{13}{2} = 0$

Here, $S = x^2 + y^2 - \frac{7}{2}x - \frac{9}{2}y - \frac{13}{2}$

$\therefore \quad S_1 = 9 + 16 - \frac{21}{2} + 18 - \frac{13}{2} = 26$

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Hence, required length of the tangent = $\sqrt{S_1} = \sqrt{26}$

Illustration 34

Find the length of the tangent drawn from any point on the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ to the circle $x^2 + y^2 + 2gx + 2fy + c_1 = 0$.

Let (x_1, y_1) be a point on the circle $x^2 + y^2 + 2gx + 2fy + c = 0$. Then,

$$x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c = 0 \quad \Rightarrow \quad x_1^2 + y_1^2 + 2gx_1 + 2fy_1 = -c \qquad \text{...(i)}$$

The length of the tangent from (x_1, y_1) to

$x^2 + y^2 + 2gx + 2fy + c_1 = 0$ is

$$= \sqrt{x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c_1} = \sqrt{-c + c_1} = \sqrt{c_1 - c}$$

Illustration 35

The distances from the origin of the centres of the circles $x^2 + y^2 + 2\lambda_i x - c^2 = 0$, $i = 1, 2, 3$ are in G.P. Prove that the lengths of the tangents drawn from any point on the circle $x^2 + y^2 = c^2$ to these circles are also in G.P.

Let, $P(x_1, y_1)$ be any point on $x^2 + y^2 = c^2$ then,

$$x_1^2 + y_1^2 = c^2 \quad \Rightarrow \quad x_1^2 + y_1^2 - c^2 = 0 \qquad \text{...(i)}$$

The centres of the circles $x^2 + y^2 + 2\lambda_i x - c^2 = 0$, $i = 1, 2, 3$ are

$C_1(-\lambda_1, 0)$, $C_2(-\lambda_2, 0)$ and $C_3(-\lambda_3, 0)$

The distances from origin to the centres C_1, C_2, C_3 are $\lambda_1, \lambda_2, \lambda_3$ respectively.

Since, these distances are in G.P., $\therefore \lambda_2^2 = \lambda_1\lambda_3$ ...(ii)

Let PT_1, PT_2, and PT_3 be the lengths of the tangents drawn from $P(x_1, y_1)$ to the three circles $x^2 + y^2 + 2\lambda_i x - c^2 = 0$, $i = 1, 2, 3$.

Then,

$PT_1 = \sqrt{x_1^2 + y_1^2 + 2\lambda_1 x_1 - c^2}$,

$PT_2 = \sqrt{x_1^2 + y_1^2 + 2\lambda_2 x_1 - c^2}$,

and $\quad PT_3 = \sqrt{x_1^2 + y_1^2 + 2\lambda_3 x_1 - c^2}$,

$\Rightarrow \quad PT_1 = \sqrt{2\lambda_1 x_1}$, $PT_2 = \sqrt{2\lambda_2 x_1}$ and $PT_3 = \sqrt{2\lambda_3 x_1}$ $\qquad [\because x_1^2 + y_1^2 = c^2]$

now, $\quad PT_2^2 = 2\lambda_2 x_1$

and, $\quad PT_1 . PT_3 = 2\sqrt{\lambda_1\lambda_3}\, x_1 . = 2\sqrt{\lambda_2^2}\, x_1 = 2\lambda_2 x_1$ $\qquad$ [Using (ii)]

$\therefore \quad PT_2^2 = PT_1 . PT_3 \Rightarrow PT_1, PT_2, PT_3$ are in G.P.

Illustration 36

Find the locus of the point the tangents drawn from which to the circles $x^2 + y^2 - 4x + 3y - 1 = 0$ and $3x^2 + 3y^2 + 5x - 7y + 2 = 0$ are equal.

Let $P(h, k)$ be the given point. Let PT_1 and PT_2 be the lengths of the tangents drawn from P to the circles $x^2 + y^2 - 4x + 3y - 1 = 0$ and $3x^2 + 3y^2 + 5x - 7y + 2 = 0$ respectively. Then,

$PT_1 = \sqrt{h^2 + k^2 - 4h + 3k - 1}$

and $\quad PT_2 = \sqrt{h^2 + k^2 + \frac{5}{3}h - \frac{7}{3}k + \frac{2}{3}}$

now, $\quad PT_1 = PT_2$ (given) $\quad \Rightarrow PT_1^2 = PT_2^2$

$\Rightarrow \quad h^2 + k^2 - 4h + 3k - 1 = h^2 + k^2 + \frac{5}{3}h - \frac{7}{3}k + \frac{2}{3}$

$\Rightarrow \quad \frac{17}{3}h - \frac{16}{3}k + \frac{5}{3} = 0 \quad \Rightarrow 17h - 16k + 5 = 0$

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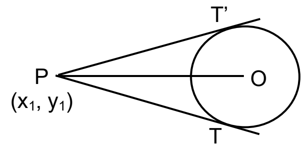
Hence, the locus of (h, k) is $17x - 16y + 5 = 0$

Pair of tangents drawn from a point to a given circle

Let (x_1, y_1) be a given point and let the equation of the circle be $x^2 + y^2 = a^2$. Then, the equation of any tangent to the circle is of the form $y = mx + a\sqrt{1+m^2}$, where m is the slope of the tangent. If this tangent passes through the given point (x_1, y_1) we must have



$$y_1 = mx_1 + a\sqrt{1+m^2}$$

$\Rightarrow \quad (y_1 - mx_1)^2 = a^2(1 + m^2)$

$\Rightarrow \quad m^2(x_1^2 - a^2) - 2m\,x_1 y_1 + y_1^2 - a^2 = 0 \qquad \ldots(i)$

The equation, being quadratic in m, given two values of m (real, coincident or imaginary) corresponding to any given values of x_1 and y_1. For each value of m we have a corresponding tangent. Hence in general two tangents can be drawn from the point (x_1, y_1) to a circle.

The tangents are real, coincident or imaginary according as the values of m obtained from (i) are real, coincident or imaginary.

or, according as Disc >, = or < 0

or, according as $4x_1^2 y_1^2 - 4(x_1^2 - a^2)(y_1^2 - a^2) >, =$ or < 0

or, according as $x_1^2 + y_1^2 - a^2 >, = \text{or} < 0$

or, according as the point P (x_1, y_1) lies outside on, or, inside the circle $x^2 + y^2 = a^2$.

Combined equation of pair of tangents

As we have seen in article that two tangents can be drawn from a point P (x_1, y_1) to a given circle. The combined equation of the pair of tangents drawn from a point (x_1, y_1) to a circle

$x^2 + y^2 + 2gx + 2fy + c = 0$ is given by

$$(x^2 + y^2 + 2gx + 2fy + c)(x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c)$$

$$= \{xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c\}^2$$

or $\quad SS_1 = T^2$.

Illustration 37

Find the equation of the pair of tangents drawn from the point (4, 3) to the circle $x^2 + y^2 - 12x + 12y + 20 = 0$

The combined equation of the tangents drawn from (4, 3) to the given circle is $SS_1 = T^2$ i.e.

$(x^2 + y^2 - 12x + 12y + 20)(16 + 9 - 48 + 36 + 20)$

$= \quad \{(4x + 3y - 6(4 + x) + 6(y + 3) + 20\}^2$

or, $\quad 33(x^2 + y^2 - 12x + 12y + 20) = (-2x + 9y + 14)^2$

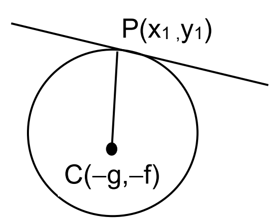
or, $\quad 29x^2 - 48y^2 + 36xy - 340x + 144y + 464 = 0$.

Normal to a circle at a given point

Normal. The normal at any point on a curve is a straight line which is perpendicular to the tangent to the curve at the point. Normal always passes through the centre of the circle.

The equation of normal at the point P (x_1, y_1) to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is

$$\frac{x - x_1}{x_1 + g} = \frac{y - y_1}{y_1 + f}$$



Proof : Centre of the circle is C(– g , –f)

Normal at P (x_1, y_1) passes through the centre C(–g,–f)

$\therefore$ using two point form, equation of normal CP is

$$y - y_1 = \frac{(y_1 + f)}{x_1 + g}(x - x_1)$$

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or $\dfrac{x-x_1}{x_1+g}=\dfrac{y-y_1}{y_1+f}$

Illustration 38

Find the equation of the normal to the circle $x^2 + y^2 - 5x + 2y - 48 = 0$ at the point (5, 6).

Centre $C \equiv \left(\dfrac{5}{2}, -1\right)$ point $P \equiv (5, 6)$

$\therefore$ Equation of normal is

$\dfrac{x-5}{5-\dfrac{5}{2}}=\dfrac{y-6}{6+1} \Rightarrow 14x - 5y - 40 = 0$

Alternative method

The equation of the tangent to the circle $x^2 + y^2 - 5x + 2y - 48 = 0$ at (5, 6) is

$5x + 6y - 5\left(\dfrac{x+5}{2}\right) + 2\left(\dfrac{y+6}{2}\right) - 48 = 0$

$\Rightarrow$ $10x + 12y - 5x - 25 + 2y + 12 - 96 = 0$ $\Rightarrow 5x + 14y - 109 = 0$

$\therefore$ Slope of the tangent $= -\dfrac{5}{14}$ $\Rightarrow$ Slope of the normal $= \dfrac{14}{5}$.

Hence, the equation of the normal at (5, 6) is

$y - 6 = (14/5)\ (x - 5)$ $\Rightarrow 14x - 5y - 40 = 0.$

Practice Assignment - II

1. Find the equation of the circle whose centre is (3, 4) and which touches the line $5x + 12y = 1$.
2. Find the equation of the circle whose centre is (3, –1) and which cut, off an intercept of length 6 from the line $2x - 5y + 18 = 0$.
3. Find the length of the chord $4x - 3y = 5$ of the circle $x^2 + y^2 + 3x - y - 10 = 0$
4. Prove that the line $y = x + a\sqrt{2}$ touches the circle $x^2 + y^2 = a^2$. Also find the point of contact.
5. Find the equations of the tangents to the circle $x^2+y^2 = 9$ which
   (i) are parallel to the line $3x + 4y - 5 = 0$
   (ii) are perpendicular to the line $2x + 3y + 7 = 0$
   (iii) make an angle of 60° with the x axis.
6. Find the equations of tangents to the circle $x^2 + y^2 - 2x - 4y - 4 = 0$ which are
   (i) parallel (ii) perpendicular
   to the line $3x - 4y - 1 = 0$.
7. **Find the equation of a circle whose centre is the point (1,–3) and which touches the line $2x - y - 4 = 0$.**
8. **Find the equation of the circle with radius $\sqrt{13}$ and touching the line $2x-3y +1 = 0$ at (1,1).**
9. **Prove that the tangent to the circle $x^2 + y^2 = 5$ at the point (1,–2) also touches the circle $x^2 + y^2 - 8x + 6y + 20 = 0$ and find its point of contact.**
10. **The extremities of a diameter of a circle are (1,2) and (3,4). Find its equation. Also find the equations of tangents which are parallel to this diameter.**
11. From the point A(0, 3) on the circle $x^2 + 4x + (y - 3)^2 = 0$, a chord AB is drawn and extended to a point M such that AM = 2 AB. Find the equation of the locus of M.
12. Find the length of the tangent drawn from the point (5, 1) to the circle $x^2 + y^2 + 6x - 4y - 3 = 0$.
13. If the length of tangent from (f,g) to the circle $x^2+y^2 = 6$ be twice the length of the tangent from the same point to the circle $x^2+ y^2 + 3x + 3y = 0$, then find the value of $f^2 + g^2 + 4f + 4g$.
14. If the lengths of the tangent from the point (1, 2) to the circles $x^2 + y^2+ x + y - 4 = 0$ and

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$3x^2 + 3y^2 - x - y - \lambda = 0$ are in the ratio 4 : 3, find the value of λ.

15. Show that if the length of the tangent from a point P to the circle $x^2+y^2 = a^2$ be four times the length of the tangent from it to the circle $(x-a)^2+y^2=a^2$ then P lies on the circle

$$15x^2+15y^2 - 32\, ax + a^2 = 0.$$

16. Find the area of the quadrilateral formed by a pair of tangents from the point (4, 5) to the circle $x^2 + y^2 - 4x - 2y - 11 = 0$ and a pair of its radii.

17. Find the equation of the pair of tangents drawn to the circle $x^2 + y^2 - 2x + 4y = 0$ from the point (0, 1)

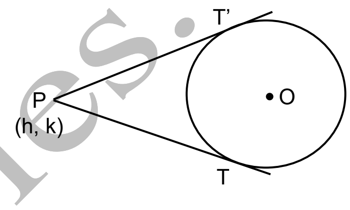
18. Find the equation of the normal to the circle $x^2 + y^2 - 2x - 4y + 3 = 0$ at the point (2, 3)

19. Find the equation of a circle of radius 3 units to which the lines $x^2 - 3xy - 3x + 9y = 0$ are normals.

20. Find the area of the triangle formed by the positive x axis and the normal and tangent to the circle $x^2 + y^2 = 4$ at $(1, \sqrt{3})$.

Director circle and its equation

DIRECTOR CIRCLE. The locus of the point of intersection of two perpendicular tangents to a given conic is known as its director circle.



Equation of director circle . The equation of the director circle to the circle $x^2 + y^2 = a^2$ is $x^2 + y^2 = 2a^2$.

The equation of any tangent to the circle $x^2 + y^2 = a^2$ is

$y = mx + a\sqrt{1+m^2}$, where m is the slope of the tangent.

If it passes through (h, k) then

$k = mh + a\sqrt{1+m^2}$

$\Rightarrow \quad (k - mh)^2 = a^2 (1 + m^2)$

$\Rightarrow \quad m^2 (h^2 - a^2) - 2mhk + (k^2 - a^2)= 0$

This equation gives two values of m, say m_1, m_2 corresponding to each value of m there is a tangent passing through (h, k).

We have, $m_1 + m_2 = \dfrac{2hk}{h^2 - a^2}$ and $m_1 m_2 = \dfrac{k^2 - a^2}{h^2 - a^2}$

If the tangents are perpendicular, then

$$m_1 m_2 = -1 \Rightarrow \frac{k^2 - a^2}{h^2 - a^2} = -1 \Rightarrow h^2 + k^2 = 2a^2.$$

Hence, locus of (h, k) is $x^2 + y^2 = 2a^2$.

Thus, the director circle of the circle is $x^2 + y^2 = a^2$ is $x^2 + y^2 = 2a^2$.

Clearly, it is a circle having the same center and radius $\sqrt{2}$ (radius of the given circle.)

Chord of contact of tangents

Chord of contact. The chord joining the points of contact of the two tangents to a conic drawn from a given point, outside it, is called the chord of contact of tangents.

Equation of chord of contact. The equation of the chord of contact of tangents drawn from a point (x_1, y_1) to the circle $x^2 + y^2 = a^2$ is $xx_1 + yy_1 = a^2$.

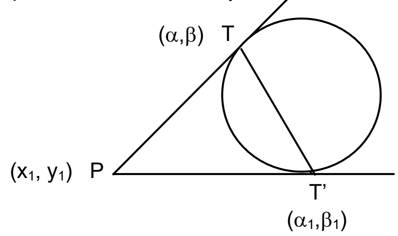
PROOF. Let, T (α, β) and T' (α_1, β_1) be the points of contact of tangents drawn from P (x_1, y_1) to $x^2 + y^2 = a^2$. The equation of tangent at T (α, β) and T' (α_1, β_1) to the circle $x^2 + y^2 = a^2$ are

$\alpha x_1 + \beta y_1 = a^2 \quad$ ...(i) $\quad$ and $\quad \alpha_1 x + \beta_1 y = a^2 \quad$ ...(ii)

Since, (i) and (ii) pass through (x_1, y_1). Therefore,

$\alpha_1 x + \beta_1 y = a^2 \quad$ and $\quad \alpha_1 x_1 + \beta_1 y_1 = a^2$

$\Rightarrow (\alpha, \beta)$ and (α_1, β_1) lie on $xx_1 + yy_1 = a^2$



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Hence, the equation of TT' i.e. the chord of contact is $xx_1 + yy_1 = a^2$.

NOTE. The equation of the chord of contact is of the same form as the equation of the tangent, but in the former point (x_1, y_1) lies outside the conic and in latter the point (x_1, y_1) is on the conic (circle).

REMARK. The equation of the chord of contact of tangents drawn from (x_1, y_1) to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is $xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$.

Illustration 39

Find the equation of the chord of contact of tangents drawn from the point (2, –3) to the circle $x^2 + y^2 + 4x - 6y - 12 = 0$

The required equation is

$$2x - 3y + 4\left(\frac{x+2}{2}\right) - 6\left(\frac{y-3}{2}\right) - 12 = 0$$

$\Rightarrow$ $2x - 3y + 2x + 4 - 3y + 9 - 12 = 0$

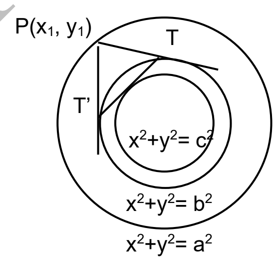
$\Rightarrow$ $4x - 6y + 1 = 0$

Illustration 40

The chord of contact of tangents drawn from a point of the circle $x^2 + y^2 = a^2$ to the circle $x^2+y^2 = b^2$ touches the circle $x^2 + y^2 = c^2$. Show that a, b, c are in G.P.

Let P (x_1, y_1) be a point on the circle $x^2 + y^2 = a^2$ then

$$x_1^2 + y_1^2 = a^2 \quad \text{...(i)}$$



The equation of the chord of contact of tangents drawn from (x_1, y_1) to the circle $x^2+y^2 = b^2$ is $xx_1 + yy_1 = b^2$ ...(ii)

This touches the circle $x^2 + y^2 = c^2$

Length of the perpendicular from (0, 0) to (ii)

= radius of $x^2 + y^2 = c^2$

$$\Rightarrow \left|\frac{0x_1 + 0y_1 - b^2}{\sqrt{x_1^2 + y_1^2}}\right| = c$$

$$\Rightarrow \frac{b^2}{\sqrt{a^2}} = c \text{ (as } x_1^2 + y_1^2 = a^2)$$

$\Rightarrow$ $b^2 = ac \Rightarrow$ a, b, c are in G.P.

Common Tangents to two Circles

Different cases of intersection of two circles:

Let the equations of the two circles be

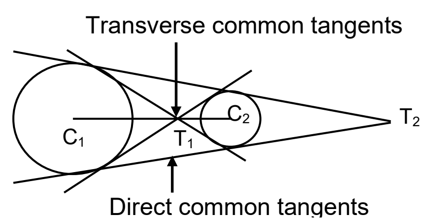
$$(x - h_1)^2 + (y - k_1)^2 = a_1^2 \quad \text{...(i)}$$

and $$(x - h_2)^2 + (y - k_2)^2 = a_2^2 \quad \text{...(ii)}$$

with centres $C_1 (h_1, k_1)$ and $C_2 (h_2, k_2)$ and radii a_1 and a_2 respectively. Then the following cases of intersection of these two circles may arise:

Case I. When $C_1C_2 > a_1 + a_2$ i.e., the distance between the centres is greater than the sum of the radii

In this case , the two circles do not intersect with each other and four common tangents can be drawn to two circles. Two of them are direct common tangents and the other two are transverse common tangents. The points of intersection of direct and transverse common tangents always lie on the line joining the centres C_1 and C_2 and



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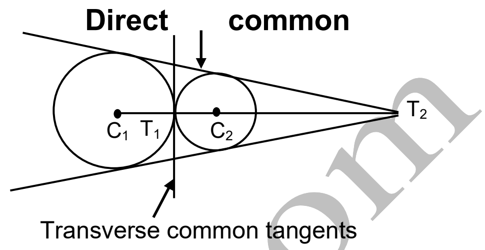
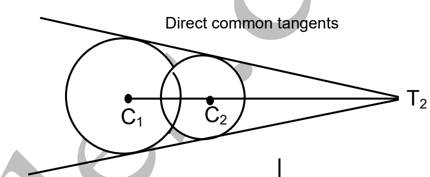
divide it externally and internally respectively in the ratio a_1 : a_2 i.e.

$$\frac{C_1T_2}{C_2T_2}=\frac{a_1}{a_2}\ \text{(externally) and}\ \frac{C_1T_1}{C_2T_1}=\frac{a_1}{a_2}\ \text{(internally)}$$

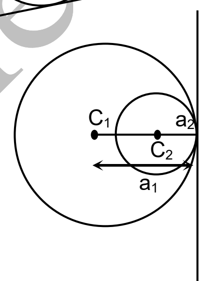
Case II. When $C_1C_2 = a_1 + a_2$ i.e. the distance between the centres is equal to the sum of the radii.
In this case, two direct tangents are real and distinct while the transverse tangents are coincident.



Case III. When $C_1C_2 < a_1 + a_2$ i.e. the distance between the centres is less than the sum of the radii.
In this case, the two direct common tangents are real while the transverse tangents are imaginary.



Case IV. When $C_1C_2 = a_1 - a_2$ i.e. the distance between the centres is equal to the difference of the radii.
In this case, two tangents are real and coincident while the other two tangents are imaginary.



Case V. When $C_1C_2 < a_1 - a_2$ i.e. the distance between the centres is less than the difference of the radii.

In this case, all the four common tangents are imaginary.

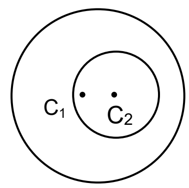


Illustration 41

If the two circles $(x-1)^2 + (y-3)^2 = r^2$ and $x^2 + y^2 - 8x + 2y + 8 = 0$ intersect in two distinct points, then prove that $2 < r < 8$.

The given equations of circles are

$$(x-1)^2 + (y-3)^2 = r^2 \qquad \text{....(1)}$$

and $x^2 + y^2 - 8x + 2y + 8 = 0$

or $(x-4)^2 + (y+1)^2 = 3^2 \qquad \text{....(2)}$

Centre and radius of (1) are C_1 (1, 3) and r and centre and radius of (2) are C_2 (4 – 1) and 3.
For circles to intersect in two points,

$|r-3| < |C_1C_2| < r+3$

or $|r-3| < \sqrt{(1-4)^2+(3+1)^2} < r+3$

or $|r-3| < 5 < r+3$

$\therefore$ $|r-3| < 5$ **(From first two relations)**

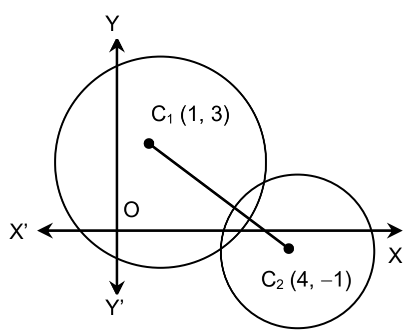
or $-5 < r-3 < 5$

or $-2 < r < 8 \qquad \text{....(3)}$

and $5 < r+3$ **(From last two relations)**

$\therefore$ $2 < r \qquad \text{.....(4)}$

From (3) and (4), we get $2 < r < 8$



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Common chord of two Circles

DEFINITION. The chord joining the points of intersection of two given circles is called their common chord.

Equation of Common Chord : The equation of the common chord of two circles

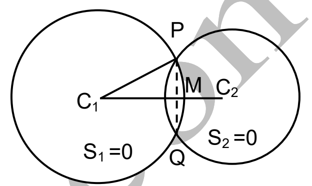
$S_1 \equiv x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$ ...(i)

and $S_2 \equiv x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$ ...(ii)

is $2x (g_1 - g_2) + 2y (f_1 - f_2) + c_1 - c_2 = 0$

i.e. $S_1 - S_2 = 0.$

PROOF. Clearly, $2x (g_1 - g_2) + 2y (f_1 - f_2) + c_1 - c_2 = 0$ is a first degree equation in x, y. So, it represents a straight line. Also, this equation is satisfied by the same values of x and y which satisfy (i) and (ii) simultaneously. Hence $S_1 - S_2 = 0$ represents the common chord of $S_1 = 0$ and $S_2 = 0$.



Length of the common chord

$|PQ| = 2\,|PM| = 2\sqrt{C_1P^2 - C_1M^2}$

Where $C_1 P$ = radius of the circle S = 0 and $C_1 M$ = length of the perpendicular from the centre C_1 to the common chord PQ.

REMARK. If the length of the common chord is zero, then the two circles touch each other and the common chord becomes the common tangent to the two circles at the common point of contact.

Illustration 42

Find the equation and the length of the common chord of the two circles

$2x^2 + 2y^2 + 7x - 5y + 2 = 0$

and $x^2 + y^2 - 4x + 8y - 18 = 0$

Writing the two circles as

$S_1 = x^2 + y^2 + \frac{7}{2}x - \frac{5}{2}y + 1 = 0$ ...(i)

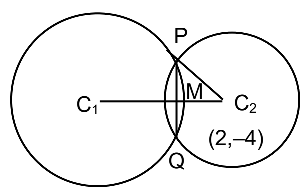
$S_2 = x^2 + y^2 - 4x + 8y - 18 = 0$ ...(ii)

and subtracting (ii) from (i), the equation of the common chord PQ is $S_1 - S_2 = 0$

i.e. $\frac{15x}{2} - \frac{21}{2}y + 19 = 0 \Rightarrow 15x - 21y + 38 = 0$ ...(iii)

The length of the perpendicular from the centre C_2 (2, – 4) of circle (ii) upon the common chord (iii) is

$C_2M = \frac{30 + 84 + 38}{\sqrt{15^2 + 21^2}} = \frac{152}{\sqrt{666}}$



Radius of the circle (ii) is, $C_2 P = \sqrt{38}$

So, the length of the common chord is given by

$|PQ| = 2\,|PM| = \sqrt{C_2P^2 - C_2M^2}$

$= 2\sqrt{38 - \left(\frac{152}{\sqrt{666}}\right)^2} = 2\sqrt{\frac{1102}{\sqrt{333}}}$

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Angle of intersection of two curves and the condition of orthogonality of two circles

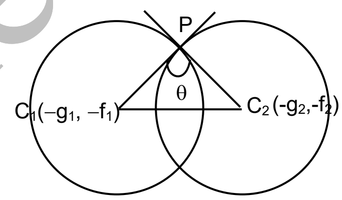
Angle of intersection of two curves If two curves C_1 and C_2 intersect at a point P and PT_1 and PT_2 be the tangents to the two curves at P. then the angle between the tangents at P is called the angle of intersection of the two curves at the point of intersection.

Orthogonal curves Two curves are said to intersect orthogonally when the tangents at the common point are at right angles.

Condition for two intersecting circles to be orthogonal

The condition for the circles $x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$ and $x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$ to intersect orthogonally is given by

$$2(g_1g_2 + f_1f_2) = c_1 + c_2$$



PROOF. Suppose the circles

$$x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0 \quad \text{...(i)}$$

and

$$x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0 \quad \text{...(ii)}$$

intersect each other orthogonally at point P. let C_1 and C_2 be the centres of the circles (i) and (ii) respectively. Let PT_1 and PT_2 be the tangents drawn to these circles at point P. Since the circles intersect orthogonally, therefore $\angle C_1PC_2 = \pi / 2$. This implies that the tangents to the two circles pass through the centres of the other circles. Now, in the right angled triangle C_1PC_2, we have

$(C_1C_2)^2 = (C_1P)^2 + (C_2P)^2$

$$\Rightarrow \quad (-g_1 + g_2)^2 + (-f_1 + f_2)^2 = \left\{\sqrt{g_1^2 + f_1^2 - c_1}\right\}^2 + \left\{\sqrt{g_2^2 + f_2^2 - c_2}\right\}^2$$

$$\Rightarrow \quad 2(g_1g_2 + f_1f_2) = c_1 + c_2$$

This is the required condition for the orthogonality of two circles.

Illustration 43

Prove that the circles $x^2 + y^2 - 6x - 8y + 24 = 0$ and $x^2 + y^2 - 8x - 6y + 24 = 0$ intersect orhtogonally.

The given circles are of the form

$x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$ and $x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$,

Where, $g_1 = -3$, $f_1 = -4$, $c_1 = 24$, $g_2 = -4$, $f_2 = -3$, and $c_2 = 24$.

$\therefore \quad 2(g_1g_2 + f_1f_2) = 2[(-3) \times (-4) + (-4) \times (-3)] = 48$

and $\quad c_1 + c_2 = 24 + 24 = 48$

$\Rightarrow \quad 2(g_1g_2 + f_1f_2) = c_1 + c_2$

Hence, the given circles intersect orthogonally.

Illustration 44

Find the equation of the circle which passes through (1, 1) and cuts orthogonally each of the circles

$x^2 + y^2 - 8x - 2y + 16 = 0$ and $\quad x^2 + y^2 - 4x - 4y - 1 = 0$.

Let the equation of the required circle be

$$x^2 + y^2 + 2gx + 2fy + c = 0 \quad \text{...(i)}$$

It passes through (1, 1) and intersects the circles

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$x^2 + y^2 - 8x - 2y + 16 = 0$ and $x^2 + y^2 - 4x - 4y - 1 = 0$ orthogonally.

$\therefore$ $1 + 1 + 2g + 2f + c = 0$

$\Rightarrow$ $2g + 2f = -c - 2$ ...(ii)

$2(-4g - f) = c + 16$

$\Rightarrow$ $-8g - 2f = c + 16$ ...(iii)

and $2(-2g - 2f) = c - 1$

$\Rightarrow$ $-4g - 4f = c - 1$ ...(iv)

Adding equation (ii) to (iii) and (iv) respectively, we obtain

$-6g = 14$ and $-2g - 2f = -3$

Solving these two equations we get $g = -\frac{7}{3}$, $f = \frac{23}{6}$.

Putting the values of g and f in (ii), we get $c = -15/3$.

Substituting the values of g, f and c in (i), we get

$$x^2 + y^2 - \frac{14}{3}x + \frac{23}{3}y - \frac{15}{3} = 0$$

$\Rightarrow$ $3(x^2 + y^2) - 14x + 23y - 15 = 0$

This is the equation of the required circle.

Family of circles

Equation of a circle through the intersection of a circle and line

The equation of a circle passing through the intersection of the circle $S = x^2 + y^2 + 2gx + 2fy + c = 0$ and the line $L = lx + my + n = 0$ is $x^2 + y^2 + 2gx + 2fy + c + \lambda(lx + my + n) = 0$ or $S + \lambda L = 0$ where λ is a constant determined by an additional condition.

Circle through the intersection of two circles

The equation of a family of circles passing through the intersection of the circles

$S_1 = x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$ and $S_2 = x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$

is $(x^2 + y^2 + 2g_1x + 2f_1y + c_1) + \lambda(x^2 + y^2 + 2g_2x + 2f_2y + c_2) = 0$, i.e., $S_1 + \lambda S_2 = 0$,

where $\lambda\ (\neq -1)$ is an arbitrary real number.

Illustration 45

Find the equation of the circle which passes through the point of intersection of the circles $x^2 + y^2 - 6x + 2y + 4 = 0$ and $x^2 + y^2 + 2x - 4y - 6 = 0$ and whose centre lies on the line $y = x$.

The equation of any circle through the intersection of the circles $S_1 = 0$ and $S_2 = 0$ is

$S_1 + \lambda S_2 = 0$.

Hence, the equation of the required circle is

$(x^2 + y^2 - 6x + 2y + 4) + \lambda(x^2 + y^2 + 2x - 4y - 6) = 0$ ...(i)

$\Rightarrow$ $x^2(1 + \lambda) + y^2(1 + \lambda) + 2x(\lambda - 3) + 2y(1 - 2\lambda) + 4 - 6\lambda = 0$

$\Rightarrow$ $x^2 + y^2 + 2x\left(\frac{\lambda - 3}{\lambda + 1}\right) + 2y\left(\frac{1 - 2\lambda}{1 + \lambda}\right) + \left(\frac{4 - 6\lambda}{1 + \lambda}\right) = 0$

the coordinates of the centre of this circle are

$$\left(-\frac{\lambda - 3}{\lambda + 1}, -\frac{1 - 2\lambda}{\lambda + 1}\right)$$

if the centre lies on $y = x$, then,

$$-\frac{1 - 2\lambda}{\lambda + 1} = -\frac{\lambda - 3}{\lambda + 1} \Rightarrow 1 - 2\lambda = \lambda - 3. \Rightarrow \lambda = \frac{4}{3}.$$

Putting the value of λ in (i), we get the required circle as

$(x^2 + y^2 - 6x + 2y + 4) + \frac{4}{3}(x^2 + y^2 + 2x - 4y - 6) = 0$

$\Rightarrow \quad 7(x^2 + y^2) - 10x - 10y - 12 = 0$

Illustration 46

Find the equation of the circle passing through the intersection of the circles $x^2 + y^2 - 4 = 0$ and $x^2 + y^2 - 2x - 4y + 4 = 0$ and touching the line $x + 2y = 0$

The equation of any circle passing through the intersection of the given circle is

$x^2 + y^2 - 2x - 4y + 4 + \lambda(x^2 + y^2 - 4) = 0$ ...(i)

$\Rightarrow \quad x^2 + y^2 - \frac{2}{\lambda+1}x - \frac{4}{\lambda+1}y + \frac{4(1-\lambda)}{\lambda+1} = 0$

The coordinates of the centre of this circle are $\left(\frac{1}{\lambda+1}, \frac{2}{\lambda+1}\right)$ and radius

$$= \sqrt{\frac{1}{(\lambda+1)^2} + \frac{4}{(\lambda+1)^2} - 4\frac{(1-\lambda)}{\lambda+1}} = \frac{\sqrt{1+4\lambda^2}}{\lambda+1}$$

If the circle (i) touches the line $x + 2y = 0$, then length of the $\perp$ from the centre = radius

$$\Rightarrow \quad \left|\frac{\frac{1}{\lambda+1} + \frac{4}{\lambda+1}}{\sqrt{1^2+2^2}}\right| = \frac{\sqrt{1+4\lambda^2}}{\lambda+1} \Rightarrow \left|\frac{5}{\sqrt{5}(\lambda+1)}\right| = \frac{\sqrt{1+4\lambda^2}}{\lambda+1}$$

$\Rightarrow \quad \sqrt{5} = \sqrt{1+4\lambda^2} \Rightarrow 1 + 4\lambda^2 = 5 \Rightarrow 4\lambda^2 = 4$

$\Rightarrow \quad \lambda = \pm 1 \Rightarrow \lambda = 1$ $[\because \lambda \neq -1]$

Putting $\lambda = 1$ in (i), we get the required circle as

$2(x^2 + y^2) - 2x - 4y = 0 \quad \Rightarrow x^2 + y^2 - x - 2y = 0$

Practice Assignment - III

1. If two tangents are drawn from a point on the circle $x^2 + y^2 = 50$ to the circle $x^2 + y^2 = 25$ then find the angle between the tangents.
2. Find the equation of the chord of contact of the tangents drawn from (1,2) to the circle $x^2 + y^2 - 2x + 4y + 7 = 0$
3. Examine if the two circles $x^2 + y^2 - 2x - 4y = 0$ and $x^2 + y^2 - 8y - 4 = 0$ touch each other externally or internally.
4. Prove that the circle $x^2 + y^2 + 2ax + c^2 = 0$ and $x^2 + y^2 + 2by + c^2 = 0$ touch each other if $\frac{1}{a^2} + \frac{1}{b^2} = \frac{1}{c^2}$
5. Without drawing a diagram to scale, show that the circle $x^2+(y-1)^2 = 16$ lies completely inside the circle $x^2+y^2-x = 26$.
6. Find the equation of the circle described on the common chord of the circles $x^2 + y^2 - 4x - 5 = 0$ and $x^2 + y^2 + 8y + 7 = 0$ as diameter.
7. Show that the circles $x^2 + y^2 - 2x - 6y - 12 = 0$ and $x^2 + y^2 + 6x + 4y - 6 = 0$ cut each other orthogonally.

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Objective Assignment

1. The equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a circle, the condition will be
(a) a = b and c = 0
(b) f = g and h = 0
(c) a = b and h = 0
(d) f = g and c = 0

2. The locus of the centre of a circle of radius 2 which rolls on the outside of the circle $x^2 + y^2 + 3x - 6y - 9 = 0$ is
(a) $x^2 + y^2 + 3x - 6y + 5 = 0$
(b) $x^2 + y^2 + 3x - 6y - 31 = 0$
(c) $x^2 + y^2 + 3x - 6y + \frac{29}{4} = 0$
(d) none of these

3. The lines $2x - 3y - 5 = 0$ and $3x - 4y = 7$ are diameters of a circle of area 154 sq. units, then the equation of the circle is
(a) $x^2 + y^2 + 2x - 2y - 62 = 0$
(b) $x^2 + y^2 + 2x - 2y - 47 = 0$
(c) $x^2 + y^2 - 2x + 2y - 47 = 0$
(d) $x^2 + y^2 - 2x + 2y - 62 = 0$

4. The equation of the circle passing through (4, 5) having the centre at (2, 2) is
(a) $x^2 + y^2 + 4x + 4y - 5 = 0$
(b) $x^2 + y^2 - 4x - 4y - 5 = 0$
(c) $x^2 + y^2 - 4x = 13$
(d) $x^2 + y^2 - 4x - 4y + 5 = 0$

5. Image of the circle $x^2 + y^2 - 6x + 8 = 0$ in the line y = x is
(a) $x^2 + y^2 + 6y + 8 = 0$
(b) $x^2 + y^2 - 6y - 8 = 0$
(c) $x^2 + y^2 - 6y + 8 = 0$
(d) none of these.

6. If (–3, 2) lies on the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ which is concentric with circle $x^2 + y^2 + 6x + 8y - 5 = 0$, then c is
(a) 11
(b) – 11
(c) 24
(d) none of these.

7. Equation of the diameter of the circle $x^2 + y^2 - 2x + 4y = 0$ which passes through the origin is
(a) x + 2y = 0
(b) x – 2y = 0
(c) 2x + y = 0
(d) 2x – y = 0.

8. The radius of the circle passing through the point P(6, 2), two of whose diameters are x + y = 6 and x + 2y = 4 is
(a) 10
(b) $2\sqrt{5}$
(c) 6
(d) 4.

9. The equation of the circle passing through (2, 1) and touching coordinates axis is
(a) $x^2 + y^2 - 2x - 2y + 1 = 0$
(b) $x^2 + y^2 + 2x + 2y + 1 = 0$
(c) $x^2 + y^2 - 2x - 2y - 1 = 0$
(d) $x^2 + y^2 + 2x + 2y - 1 = 0$

10. The circle $x^2 + y^2 + 4x - 7y + 12 = 0$ cuts an intercept on y – axis of length
(a) 3
(b) 4
(c) 7
(d) 1

11. Equation of the circle through origin which cuts intercepts of lengths a and b on axes is
(a) $x^2 + y^2 + ax + by = 0$
(b) $x^2 + y^2 - ax - by = 0$
(c) $x^2 + y^2 + bx + ay = 0$
(d) none of these

12. A line is drawn through a fixed point P (α, β) to cut the circle $x^2 + y^2 = r^2$ at A and B. Then PA.PB is equal to
(a) $(\alpha + \beta)^2 - r^2$
(b) $\alpha^2 + \beta^2 - r^2$
(c) $(\alpha - \beta)^2 + r^2$
(d) none of these.

13. The straight line y = mx + c cuts the circle $x^2 + y^2 = a^2$ in real points if,
(a) $\sqrt{a^2(1+m^2)} < c$
(b) $\sqrt{a^2(1-m^2)} < c$
(c) $\sqrt{a^2(1+m^2)} > c$
(d) $\sqrt{a^2(1-m^2)} > c$

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14. Equation of the circle which touches $3x + 4y = 7$ and passes through $(1, -2)$ and $(4, -3)$ is

(a) $x^2 + y^2 - 94x + 18y + 55 = 0$
(b) $15x^2 + 15y^2 - 94x + 18y + 55 = 0$
(c) $15x^2 + 15y^2 + 94x + 18y + 55 = 0$
(d) $x^2 + y^2 - 94x - 18y + 55 = 0$

15. Centre of a circle is $(2, 3)$. If the line $x + y = 1$, touches, its equation is

(a) $x^2 + y^2 - 4x - 6y + 4 = 0$
(b) $x^2 + y^2 - 4x - 6y + 5 = 0$
(c) $x^2 + y^2 - 4x - 6y - 5 = 0$
(d) none of these.

16. The equation of the circle which has a tangent $2x - y - 1 = 0$ at $(3, 5)$ on it and with the centre on $x + y = 5$, is

(a) $x^2 + y^2 + 6x - 16y + 28 = 0$
(b) $x^2 + y^2 - 6x + 16y - 28 = 0$
(c) $x^2 + y^2 + 6x + 6y - 28 = 0$
(d) $x^2 + y^2 - 6x - 6y - 28 = 0$

17. The locus of the point (h, k) for which the line $hx + ky = 1$ touches the circle $x^2 + y^2 = 4$ is

(a) $x^2 + y^2 = \frac{1}{2}$
(b) $x^2 + y^2 = \frac{1}{4}$
(c) $x^2 + y^2 = 1$
(d) $x^2 + y^2 = 2$

18. The lines $3x - 4y + 4 = 0$ and $6x - 8y - 7 = 0$ are tangent to the same circle. The radius of the circle is

(a) 1/2
(b) 3/4
(c) 3/2
(d) none of these.

19. The radius of the circle touching the straight lines $x - 2y - 1 = 0$ and $3x - 6y + 7 = 0$ is

(a) $\frac{3}{\sqrt{5}}$
(b) $\frac{\sqrt{5}}{3}$
(c) $\sqrt{5}$
(d) $\frac{1}{\sqrt{2}}$

20. The circle $x^2 + y^2 - 6x - 10y + c = 0$ does not intersect or touch any axis of coordinates and the point $(1, 4)$ lies inside the circle. Then the range of possible values of c is given by

(a) $25 < c < 29$
(b) $c > 9$
(c) $c > 25$
(d) $c < 29$

21. The possible values of p for which the line $x \cos \alpha + y \sin \alpha = p$ is a tangent to the circle $x^2 + y^2 - 2qx \cos \alpha - 2qy \sin \alpha = 0$, is / are

(a) 0 and q
(b) q and 2q
(c) 0 and 2q
(d) q

22. The locus of a point such that tangents drawn from it to the circle $x^2 + y^2 = a^2$ are at 120^o is :

(a) $4x^2 + 4y^2 = 3a^2$
(b) $3x^2 + 3y^2 = 4a^2$
(c) $x^2 + y^2 = \frac{2a^2}{3}$
(d) none of these

23. From a point $P(2, 2\sqrt{2})$ tangents PQ and PR are drawn to the circle $x^2 + y^2 = a^2$. If QR is touching the circle $x^2 + y^2 = 3$, then the value of a is

(a) 4
(b) $\sqrt{6}$
(c) $2\sqrt{3}$
(d) $3\sqrt{2}$

24. Area of the circle in which a chord of length $\sqrt{2}$ makes an angle $\frac{\pi}{2}$ at the centre

(a) $\frac{\pi}{2}$
(b) 2π
(c) π
(d) $\frac{\pi}{4}$

25. If the straight line $ax + by = 2$; $a, b \neq 0$, touches the circle $x^2 + y^2 - 2x = 3$ and is normal to the circle $x^2 + y^2 - 4y = 6$, then the values of a and b are

(a) $a = 1, b = 2$
(b) $a = 1, b = -1$
(c) $a = -4/3, b = 1$
(d) None of these

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26. The number of common tangents to the circles $x^2 + y^2 = 4$ and $x^2 + y^2 - 6x - 8y = 24$ is

(a) 0 (b) 1

(c) 3 (d) 4

27. The equation of the circle described on the common chord of the circles $x^2 + y^2 - 8x + y - 15 = 0$ and $x^2 + y^2 - 4x + 4y - 42 = 0$ as diameter is

(a) $x^2 + y^2 - x + 2y + 4 = 0$

(b) $x^2 + y^2 + 10x - 2y - 12 = 0$

(c) $x^2 + y^2 - 12x - 2y + 12 = 0$

(d) $x^2 + y^2 - 5x + 3y + 7 = 0$

28. Length of the common chord of the two circles $x^2 + y^2 - 2x - 4y + 1 = 0$ and $x^2 + y^2 - 4x - 2y + 1 = 0$ is

(a) $\sqrt{15}$ (b) $\sqrt{14}$

(c) $\sqrt{13}$ (d) none of these

29. Tangents drawn from the point P(1, 8) to the circle $x^2 + y^2 - 6x - 4y - 11 = 0$ touch the circle at points A and B. The equation of the circumcircle of the triangle PAB is

(a) $x^2 + y^2 + 4x - 6y + 19 = 0$

(b) $x^2 + y^2 - 4x - 10y + 19 = 0$

(c) $x^2 + y^2 - 2x + 6y - 29 = 0$

(d) None of these

30. If the circles $x^2 + y^2 = 9$ and $x^2 + y^2 + 2ax + 2y + 1 = 0$ touch each other, then a is

(a) $-\frac{4}{3}, \frac{4}{3}$ (b) 0

(c) 1 (d) $\frac{5}{3}, -\frac{5}{3}$

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Answers

PRACITCE ASSIGNMENT– I

1. (i) $x^2 + y^2 - 4y = 0$ (ii) $x^2 + y^2 + 4x - 6y - 3 = 0$
   (iii) $36x^2 + 36y^2 - 36x - 18y + 11 = 0$ (iv) $x^2 + y^2 - 2x - 2y = 0$
   (v) $x^2 + y^2 + 2ax + 2by + 2b^2 = 0$
2. (i) c(–5, 3), r = 6 (ii) c(2, 4), r = $\sqrt{65}$ (iii) c(4, –5), r = $\sqrt{53}$
   (iv) c$\left(\frac{1}{4},0\right)$; r = $\frac{1}{4}$ (v) c $\left(\frac{2}{3},0\right)$, r = $\frac{\sqrt{7}}{3}$
3. –
4. $\frac{\sqrt{23}}{2}$
5. R
6. $x^2 + y^2 - 8x + 6y - 27 = 0$
7. $\frac{3\sqrt{3}}{4}(g^2 + f^2 - c)$
8. $x^2 + y^2 - 2x - 4y - 20 = 0$
9. $x^2 + y^2 - 16x + 4y - 32 = 0$
10. $x^2 + y^2 + 6x + 6y + 9 = 0$ or $x^2 + y^2 - 2x + 2y + 1 = 0$
11. $x^2 + y^2 + 2ax + 2py - (b^2 + q^2) = 0$
12. $x^2 + y^2 - 4x - 6y + 11 = 0$
13. $15x^2 + 15y^2 - 94x + 18y + 55 = 0$
14. $x^2 + y^2 - 7x + 5y - 14 = 0$
15. $x^2 + y^2 + 4x - 21 = 0$ or $x^2 + y^2 - 12x + 11 = 0$
16. $x^2 + y^2 - 4x - 3y = 0$
17. $(x - 15)^2 + (y + 15)^2 = 225$ or $(x - 3)^2 + (y + 3)^2 = 9$
18. $(x - 1 - 2\pi)^2 + (y - 1)^2 = 1$
19. $x^2 + y^2 - 5x - y + 4 = 0$
20. $x^2 + y^2 - x - y = 0$
21. $4\sqrt{13}$
22. $x^2 + y^2 \pm 10x + 6y + 9 = 0$
23. $x^2 + y^2 \pm ax \pm by = 0$
24. Inside
25. inside
26. $a \in (-3\sqrt{2}, -\sqrt{3}) \cup (\sqrt{3}, 3\sqrt{2})$
27. $x + \frac{1}{2} = \cos\theta,\ y + \frac{\sqrt{3}}{2} = \sin\theta,\ 0 \le \theta < 2\pi$
28. $x^2 + y^2 - 4x + 2y - 11 = 0$
29. $x = \frac{-p}{2} + \frac{p}{\sqrt{2}}\cos\theta,\ y = \frac{-p}{2} + \frac{p}{\sqrt{2}}\sin\theta,\ 0 \le \theta < 2\pi$
30. 15 , 5

PRACITCE ASSIGNMENT– II

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1. $x^2 + y^2 - 6x - 8y + \frac{381}{169} = 0$
2. $x^2 + y^2 - 6x + 2y - 28 = 0$
3. 5 units
4. $\left(\frac{-a}{\sqrt{2}}, \frac{a}{\sqrt{2}}\right)$
5. (i) $3x + 4y \pm 15 = 0$
   (ii) $3x - 2y \pm 3\sqrt{13} = 0$
   (iii) $\sqrt{3}x - y \pm 6 = 0$
6. (i) $3x - 4y + 20 = 0, 3x - 4y - 10 = 0$
   (ii) $4x + 3y + 5 = 0, 4x + 3y - 25 = 0$
7. $5x^2 + 5y^2 - 10x + 30y + 49 = 0$
8. $(x - 3)^2 + (y + 2)^2 = 13$ or $(x + 1)^2 + (y - 4)^2 = 13$
9. $(3, -1)$
10. $x^2 + y^2 - 4x - 6y + 11 = 0; y = x + 3, y = x - 1$
11. $x^2 + y^2 + 8x - 6y + 9 = 0$
12. 7 units
13. -2
14. $\lambda = \frac{21}{4}$
15. –
16. 8 square units
17. $2x - y + 1 = 0, x + 2y - 2 = 0$
18. $x - y + 1 = 0$
19. $x^2 + y^2 - 6x - 2y + 1 = 0$
20. $2\sqrt{3}$ sq. units

PRACITCE ASSIGNMENT– III

1. $90°$
2. $2y + 5 = 0$
3. Internally
6. $x^2 + y^2 - 2x + 4y + 1 = 0$

Objective Assignment

1.	C	11.	B	21.	C
2.	B	12.	B	22.	B
3.	C	13.	C	23.	B
4.	B	14.	B	24.	C
5.	C	15.	B	25.	C
6.	B	16.	A	26	B
7.	C	17.	B	27	C
8.	B	18.	B	28	B
9.	A	19.	B	29	B
10.	D	20.	A	30	A

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